Chapter 1
Numbers and the Base-Ten System

1.1 The Counting Numbers

1. Answers will vary. For example, when connecting the counting numbers as a list view of numbers with the number of objects in a set view of numbers, a child must learn to associate each number in the list in a one to one correspondence with each object in the set, starting with one. Also, the child must be able to learn that the last number from the list, used to connect with the last object in the set, is the number of objects in the set.

2. Yes, there is a better way to respond. For instance, you could group the beads into sets of 10 beads in each group. Then you would have 3 groups of 10 beads in each group and there would be 5 left over beads. This grouping would facilitate a discussion about place value and allow the conversation to focus on 3 tens.

3. a. You could group the beads into sets of 10 beads in each group. Then you would have 4 groups of 10 beads in each group and there would be 7 left over beads. Using the place value system of representing numbers, 4 tens and 7 ones is 47. Figure 1.1 shows a simple math drawing that could be drawn.

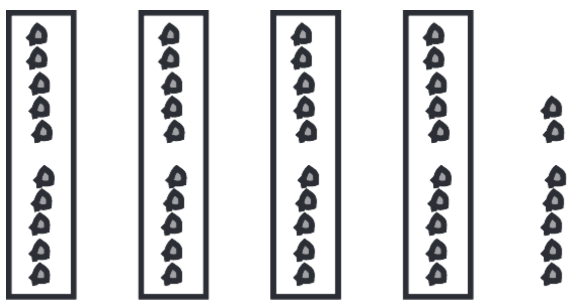


Figure 1.1: Representation of 47

   b. You could bag the toothpicks into sets of 10 toothpicks in each bag. Then when you get 10 bags of 10 toothpicks in each, you could bundle, with a rubber band, 10 bags of 10 toothpicks to make sets of 100 toothpicks in each bundle. Then you would have 3 bundles of 100 toothpicks in each bundle (or 3 hundreds) and you would have 2 bags of 10 toothpicks in each bag (or 2 tens) and there would be 8 left over toothpicks. Using the place value system of representing numbers, 3 hundreds, 2 tens, and 8 ones is 328. Figure 1.2 shows a simple math drawing that could be drawn.

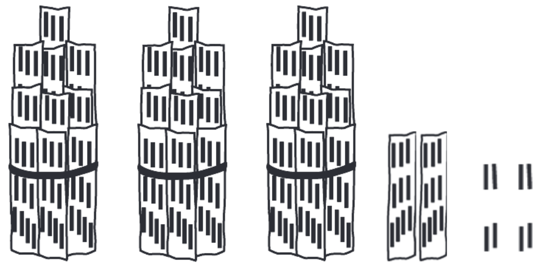


Figure 1.2: Representation of 328

c. You could bag the toothpicks into sets of 10 toothpicks in each bag. Then when you get 10 bags of 10 toothpicks in each, you could bundle, with a rubber band, 10 bags of 10 toothpicks to make sets of 100 toothpicks in each bundle. Then when you have 10 bundles of 100 toothpicks, you could get a giant gallon sized plastic bag and put them into it and group these 10 sets of 100 toothpicks into 1 set of 1000 toothpicks. Using the place value system of representing numbers, 1 thousand is represented as 1000. Figure 1.3 shows a simple math drawing that could be drawn.

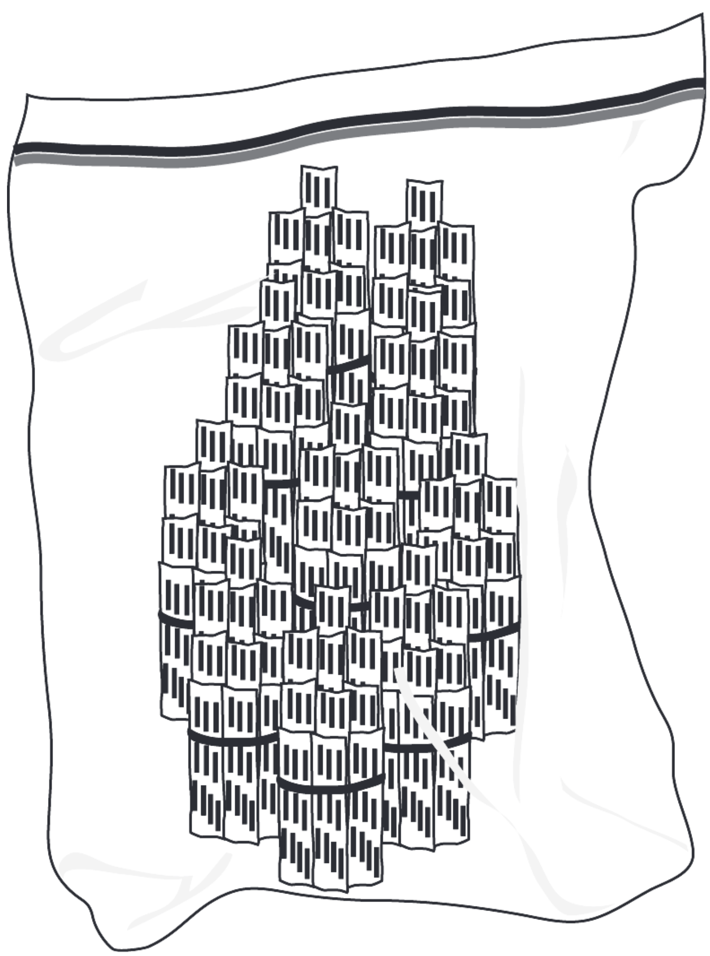


Figure 1.3: Representation of 1000

4.

a. Let's say you have a relatively unorganized collection of 62 toothpicks as shown below in Figure 1.4

Figure 1.4: Representation of 62 toothpicks

You could bag the toothpicks into sets of 10 toothpicks in each bag
See Figure 1.5.

Figure 1.5: Math drawing of 10 ones regrouped as 1 ten.

You could continue doing that with all 62 toothpicks, that is, you can continue collecting sets of ten toothpicks and regrouping them into tens until you can no longer do that. When you are done, you will be able to count the number of tens. If you look at the number 62, the digit in the tens place is a 6 and that corresponds with the number of tens that you find. Similarly, the digit is ones place is 2 and we have 2 single toothpicks not in bundles of ten. See Figure 1.6.

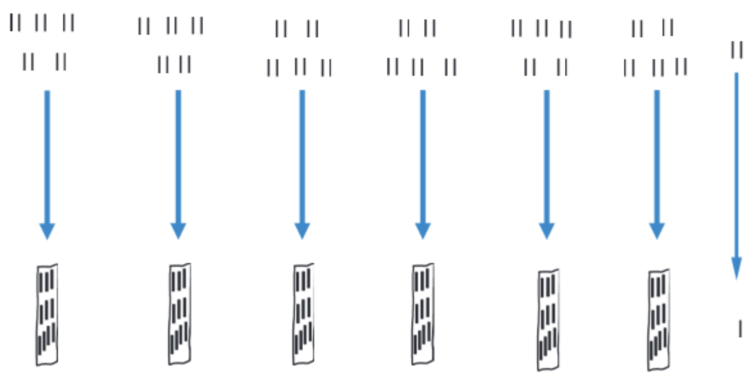


Figure 1.6: 62 represented in base-ten bundles.

b. The 6 in the number 62 could represent 6 tens, as shown in Figure 1.6. If we took the bundles apart, we would have 60 ones so 6 can stand for that as well. In summary the 6 stands for 6 tens or 60 ones.

5.

a.

You could bag the toothpicks into sets of 10 toothpicks in each bag. Then when you get 10 bags of 10 toothpicks in each, you could bundle, with a rubber band, 10 bags of 10 toothpicks to make sets of 100 toothpicks in each bundle. Then you would have 3 bundles of 100 toothpicks in each bundle (or 3 hundreds) and you would have 2 bags of 10 toothpicks in each bag (or 2 tens) and there would be 8 left over toothpicks. Using the place value system of representing numbers, 3 hundreds, 5 tens, and 8 ones is 328. Figure 1.7 shows a simple math drawing that could be drawn.

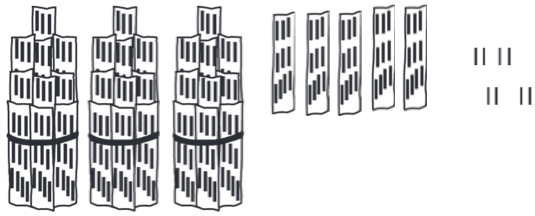


Figure 1.7: Representation of 358

b. The 5 in the number 358 could represent 5 tens, as shown in Figure 1.7. If we took the bundles apart, we would have 50 ones so 5 can stand for that as well. In summary the 5 stands for 5 tens or 50 ones.

c. The 3 in the number 358 could represent 3 hundreds, as shown in Figure 1.7. If we took the bundles apart, we would have 30 tens so 3 can stand for that as well. If we took apart the ten bundles, we would have 300 single toothpicks, so 3 could stand for that as well. In summary, the 3 stands for 3 hundreds, 30 tens or 300 ones.

6.

For instance, if you we are using Popsicle sticks, you could get a large collection (say three-hundred fifty-six sticks) in an unorganized pile. Then you could ask the students that are learning about place value to start counting them. After a bit, someone (you or some of the students) will likely start to group the Popsicle sticks into piles of equal size. You could then count the Popsicle sticks faster by bundling groups of 10 together. Perhaps you might bundle these groups together physically by tying a twist tie around each set of 10 Popsicle sticks.

After a while of doing this, you will have lots of bundled sets of 10 Popsicle sticks. At this stage, you could take 10 sets of twist-tied sets of 10 Popsicle sticks and put them in to a gallon sized plastic bag to make groups of 100 Popsicle sticks (consisting of 10 sets of 10 bundled Popsicle sticks). As you continue making twist-tied bundles of 10 and baggies of 100, you eventually will use up all of the Popsicle sticks. When this is done, you would end up with 3 baggies (or 3 hundreds) and 5 twist-tied bundles (or 5 tens) and 6 left over Popsicle sticks. Now you could point out that your baggies, bundles, and individual Popsicle sticks correspond directly with the base-ten representation of three hundred fifty-six Popsicle sticks (or 356). Since the digit in each place value is representing a count of units that consist of 10 bundles of units that are represented in the digit to the right, we see that each place value represents a number that is 10 times greater than the place value to its immediate right. For example, when we count the farthest left place value in this number (365), we count groups of hundred (3 baggies). Since each baggie is made up of 10 bundles, we note that the place value immediately to right of the hundreds place is the place value where we are counting the bundles (or tens).

7. Young children must learn several key ideas about place value and overcome some linguistic hurdles to learn how to count in the base-ten structure. They must understand

the key role that 10 plays in our base-ten structure or in representing numbers. They must understand how the location affects the value of the number or the unit of the digit in that particular location. They must overcome linguistic difficulties inherent to how we say numbers in English, especially the anomalies like eleven or the inconsistent order of twenty-two (2 tens and two ones) and nineteen (one nine and one ten).

8. The number 1001 looks like 100 and 1 put together. Calling it "one hundred one" makes sense w/o understanding the structure of our number system. See Figure 1.8. Each small block represents 1.

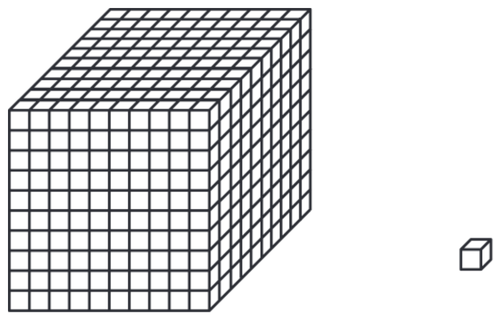


Figure 1.8: Base-Ten representation of 1001

9. If we count what we've got in the math drawing, we see 17 individual toothpicks and 15 bags of 10 toothpicks in each bag. Naively, we might write this as 1517 toothpicks, which would be misleading since as written it would represent one-thousand five-hundred seventeen toothpicks.

Since our place value system can only represent up to 9 of any particular place value, we have to regroup when we have more than 9 of a particular place value (or base-ten unit). In terms of toothpicks, this means that since 17 is greater than 9, we have enough individual toothpicks to regroup into one more baggie of 10. This gives us 7 left over toothpicks but now 16 bags. Similarly, we also have enough bags to regroup (or bundle) them with a rubber band into one group of 100 toothpicks. This gives us 1 bundle of 100 toothpicks (or 1 hundred), 6 bags of ten toothpicks (or 6 tens) and 7 individual toothpicks. See Figure 1.9 for what Figure 1.15 (in the regular text) would look like once you've regrouped the numbers in a way that corresponds to the base-ten structure.

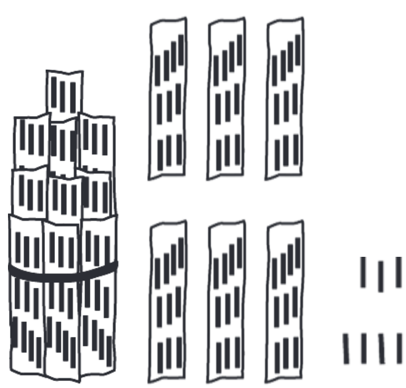


Figure 1.9: Representation of 167

10. Answers will vary. In the base-ten structure, the digits represent different values of objects. The place value is integrally related to the value that any particular digit represents. The number ten plays a vital role in the system and is the basis of the value of each place. The base-ten structure is much easier to represent large numbers than more primitive ways of representing numbers. However, the base-ten structure is not as intuitive and is harder to learn than more primitive systems, such as a simple tally mark system.

11.

a. You cannot tell where to plot 5 on the given number line. Without another point for reference, you cannot determine how much space on the given number line that a unit represents or that some other quantity of units represent.

b. You cannot plot 100 on the given number line. The reasoning is the same as the answer in part a above.

c. You cannot plot N+1 on the given number line. Without knowing what N represents, you don't know how many partitions to make between 0 and N. Those partition sizes would be each 1 unit, so you cannot determine where one more unit past N would be.

12.

a. See Figure 1.10. In the first number line I first used the larger tick marks to represent 400 each. Then I counted up to 800 and then 1200 using these sized tick marks. Realizing that 900 wasn't going to fall perfectly on a 400 tick mark, I partitioned the 400 tick marks spaces into 4 smaller spaces and then each shorter tick mark represented 100. I then counted one more shorter tick mark past 800 to get to 900.

b. See Figure 1.10. For the second number line, we partitioned the space between 0 and 300 into 3 equal spaces so between taller tick marks represents 100. Then we partitioned each of these spaces representing 100 into 2 small spaces representing 50. I counted up to 200 with large tick marks and then over one more smaller space to 250.

c. See Figure 1.10. For the last number line, I let the spaces between taller tick marks represent 2000. I went up to 6,000. Next, since it takes ten 200s to make 2,000 and the gap between 6,000 and 8,000 is 2,000, then I made 10 spaces between 6,000 and 8,000 and plotted 6,200 one of these spaces above 6,000.

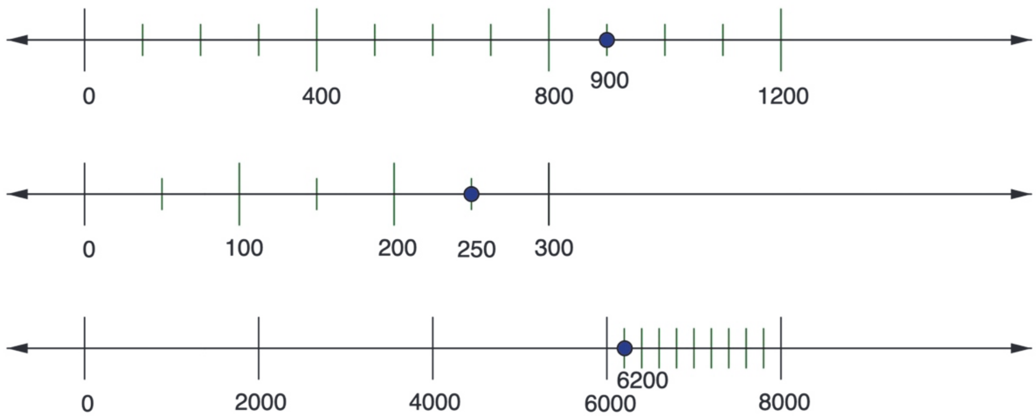


Figure 1.10: Number lines

13.

a. The next two units in base-two after 16 are 32 and 64.

b. See Table 1.1

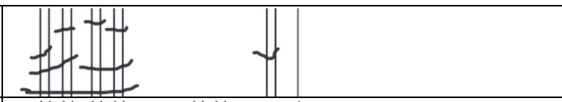
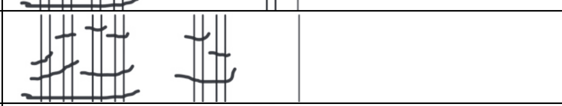
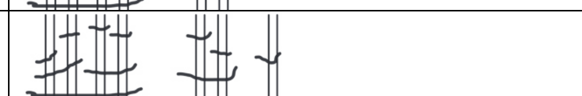
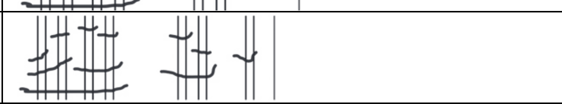
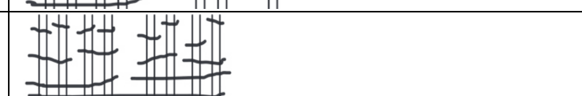
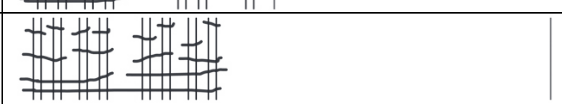
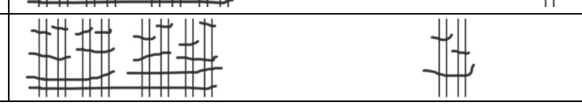
11		12	
13		14	
15		16	
17		18	
19		20	

Table 1.1: 11 through 20 in base-two representation

c. $11 = 1011_2$
$12 = 1100_2$
$13 = 1101_2$
$14 = 1110_2$
$15 = 1111_2$
$16 = 10000_2$
$17 = 10001_2$
$18 = 10010_2$
$19 = 10011_2$
$20 = 10100_2$

d. See Figure 1.11. $35 = 100011_2$.

Figure 1.11: 35 written in base-two.

e. See Figure 1.12. $45 = 101101_2$.

Figure 1.12: 45 written in base-two.

14.

a. The next two units in base-three after 3 are 9 and 27.

b. See Table 1.2.

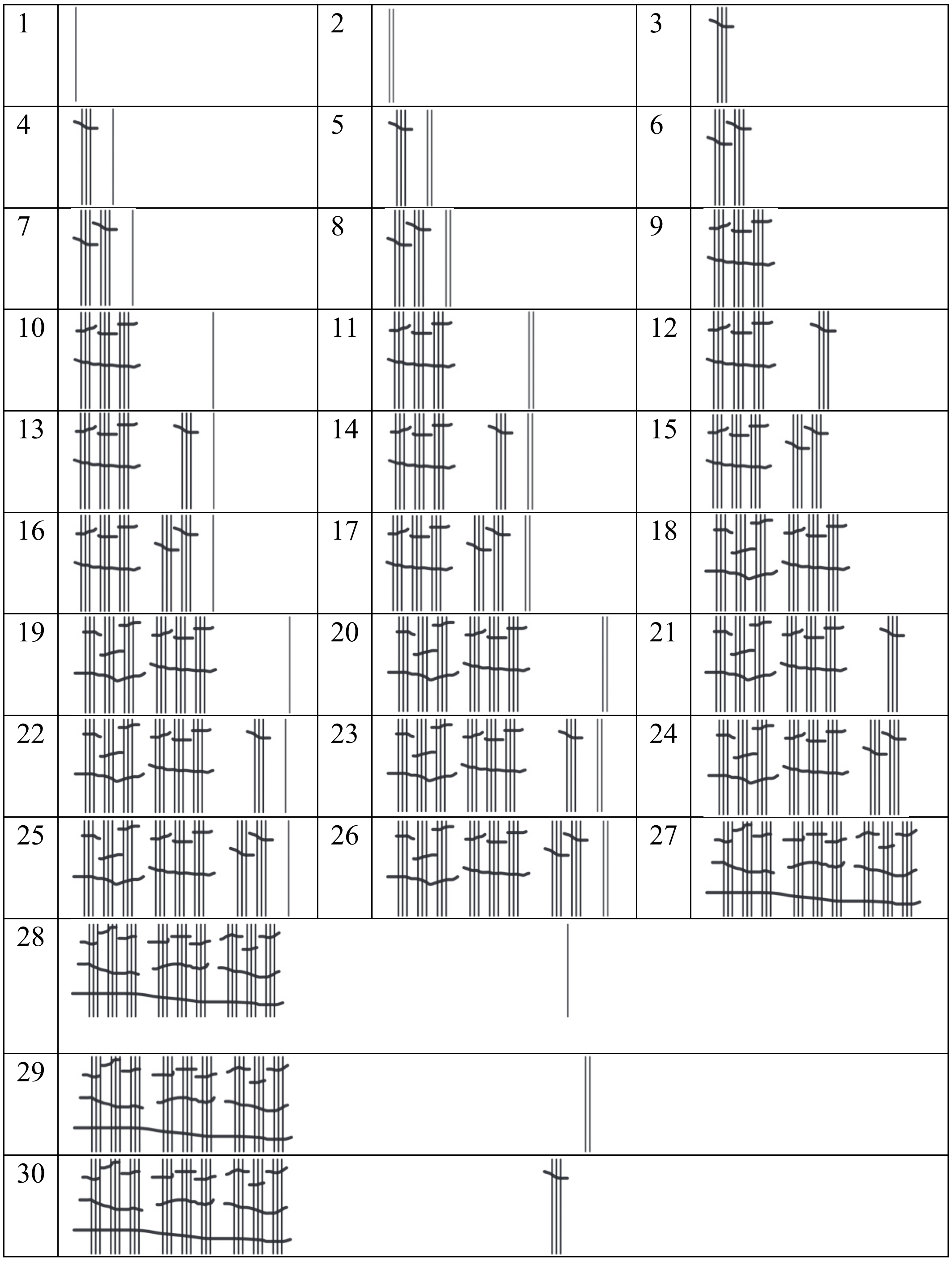


Table 1.2: Numbers 1 to 30 in base-three math drawings

c. See Table 1.3.

$1 = 1_3$	$11 = 102_3$	$21 = 210_3$
$2 = 2_3$	$12 = 110_3$	$22= 211_3$
$3 = 10_3$	$13 = 111_3$	$23 = 212_3$
$4 = 11_3$	$14 = 112_3$	$24 = 220_3$
$5 = 12_3$	$15 = 120_3$	$25 = 221_3$
$6 = 20_3$	$16 = 121_3$	$26 = 222_3$
$7 = 21_3$	$17 = 122_3$	$27 = 1000_3$
$8 = 22_3$	$18 = 200_3$	$28 = 1001_3$
$9 = 100_3$	$19 = 201_3$	$29 = 1002_3$
$10 = 101_3$	$20 = 202_3$	$30 = 1010_3$

Table 1.3: Numbers 1 to 30 in base-three

d. See Figure 1.13. $45 = 120_3$.

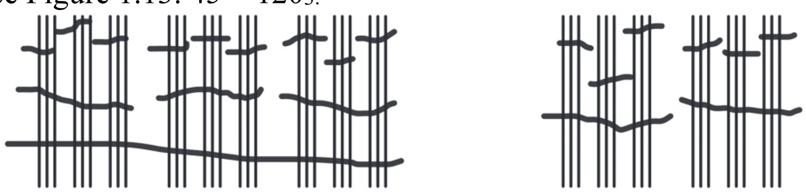


Figure 1.13: 45 written in base-three.

15.

a. The next two units in base-eight after 8 are 64 and 256.

b. See Table 1.4.

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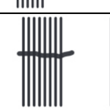
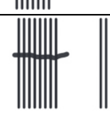
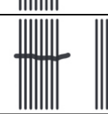
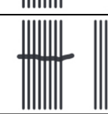
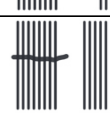
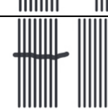
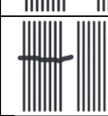
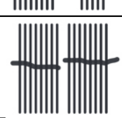
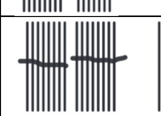
4		5		6	
7		8		9	
10		11		12	
13		14		15	
16		17		18	
19		20			

Table 1.4: Numbers 1 to 20 in base-eight math drawings

c. See Table 1.5.

$1 = 1_8$	$5 = 5_8$	$9 = 11_8$	$13 = 15_8$	$17 = 21_8$
$2 = 2_8$	$6 = 6_8$	$10 = 12_8$	$14 = 16_8$	$18 = 22_8$
$3 = 3_8$	$7 = 7_8$	$11 = 13_8$	$15 = 17_8$	$19 = 23_8$
$4 = 4_8$	$8 = 10_8$	$12 = 14_8$	$16 = 20_8$	$20 = 24_8$

Table 1.5: Numbers 1 to 20 in base-eight

d. See Figure 1.14. $100 = 144_8$.

Figure 1.14: 100 written in base-eight.

16. Answers will vary. For example, they could use a photocopier and shrink the poster to a small size (but where the dots are still distinguishable hopefully), then they could make 999 copies of this small sized poster. This would work since each of the 1000 shrunk down posters contain 1 million dots and 1,000,000 x 1,000 is 1 billion. It would probably be doable; however, one might need to find a special photocopier (like one that photocopies large maps) if the poster is large enough. Or for a more green approach, one could try to have each student try to draw a portion of the dots themselves. For example, if there were 25 students in the class then each student would need to represent 40 million dots. This would likely be not very feasible because if you worked every day on this for 4 weeks (for a total of 5x4 or 20 days) each student would still need to draw $40 \div 20$ million or 2 million dots a day. If they worked for 50 minutes on it, that would be $2{,}000{,}000 \div 50$ or 40,000 dots every minute. Even if they could do this, the class would likely revolt after two or three days of drawing dots day after day.

1.2 Decimals and Negative Numbers

1.

a. It represents one tenth. Since it takes ten of those to make one, then each long strip is one tenth of a unit or one tenth of one.

b. It represents one hundredth. Since it takes one hundred of those to make one, then each small square is one hundredth of a unit or one hundredth of one.

c. The figure represents 1.74. As described above and in the directions, each large square represents one. In the figure, there is one large square so there is 1 one. Similarly, there are 7 tenths (or 7 long strips of small squares) and 4 hundredths (or 4 small squares).

d. See Figure 1.15. As described above and in the directions, each large square represents one. In the figure, there are 3 large squares, so we have 3 ones. Similarly, there are 2 tenths (or 2 long strips of small squares) and 5 hundredths (or 5 small squares).