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Matrix Analysis

Exercises 1.3.3

- **1(a)** Yes, as the three vectors are linearly independent and span three-dimensional space.

1(b) No, since they are linearly dependent

$$\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1(c) No, do not span three-dimensional space. Note, they are also linearly dependent.

- **2** Transformation matrix is

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotates the $(\mathbf{e}_1, \mathbf{e}_2)$ plane through $\pi/4$ radians about the $\mathbf{e}_3$ axis.

- **3** By checking axioms (a)–(h) on p. 10 it is readily shown that all cubics $ax^3 + bx^2 + cx + d$ form a vector space. Note that the space is four dimensional.

3(a) All cubics can be written in the form

$$ax^3 + bx^2 + cx + d$$

and $\{1, x, x^2, x^3\}$ are a linearly independent set spanning four-dimensional space. Thus, it is an appropriate basis.

3(b) No, does not span the required four-dimensional space. Thus a general cubic cannot be written as a linear combination of

$$(1-x),\ (1+x),\ (1-x^3),\ (1+x^3)$$

as no term in x^2 is present.

3(c) Yes as linearly independent set spanning the four-dimensional space

$$a(1-x)+b(1+x)+c(x^2-x^3)+d(x^2+x^3)$$

$$=(a+b)+(b-a)x+(c+a)x^2+(d-c)x^3$$

$$\equiv \alpha+\beta x+\gamma x^2+\delta x^3$$

3(d) Yes as a linear independent set spanning the four-dimensional space

$$a(x-x^2)+b(x+x^2)+c(1-x^3)+d(1+x^3)$$

$$=(a+b)+(b-a)x+(c+d)x^2+(d-c)x^3$$

$$\equiv \alpha+\beta x+\gamma x^2+\delta x^3$$

3(e) No not linearly independent set as

$$(4x^3+1)=(3x^2+4x^3)-(3x^2+2x)+(1+2x)$$

▪ **4** $x+2x^3$, $2x-3x^5$, $x+x^3$ form a linearly independent set and form a basis for all polynomials of the form $\alpha+\beta x^3+\gamma x^5$. Thus, S is the space of all odd quadratic polynomials. It has dimension 3.

Exercises 1.4.3

■ **5(a)** Characteristic polynomial is $\lambda^3 - p_1\lambda^2 - p_2\lambda - p_3$ with
$p_1 =$ trace $\mathbf{A} = 12$

$$\mathbf{B}_1 = \mathbf{A} - 12\mathbf{I} = \begin{bmatrix} -9 & 2 & 1 \\ 4 & -7 & -1 \\ 2 & 3 & -8 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}\ \mathbf{B}_1 = \begin{bmatrix} -17 & -5 & -7 \\ -18 & -30 & 7 \\ 2 & -5 & -33 \end{bmatrix}$$

$p_2 = \dfrac{1}{2}$ trace $\mathbf{A}_2 = -40$

$$\mathbf{B}_2 = \mathbf{A}_2 + 40\mathbf{I} = \begin{bmatrix} 23 & -5 & -7 \\ -18 & 10 & 7 \\ 2 & -5 & 7 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}\ \mathbf{B}_2 = \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$p_3 = \dfrac{1}{3}$ trace $\mathbf{A}_3 = 35$

Thus, characteristic polynomial is

$$\lambda^3 - 12\lambda^2 + 40\lambda - 35$$

Note that $\mathbf{B}_3 = \mathbf{A}_3 - 35\mathbf{I} = 0$ confirming check.

5(b) Characteristic polynomial is $\lambda^4 - p_1\lambda^3 - p_2\lambda^2 - p_3\lambda - p_4$ with
$p_1 =$ trace $\mathbf{A} = 4$

$$\mathbf{B}_1 = \mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & -3 & 1 & 0 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}\ \mathbf{B}_1 = \begin{bmatrix} -3 & 4 & 0 & -3 \\ -1 & -2 & -2 & 1 \\ 2 & 0 & -2 & -5 \\ -3 & -3 & -1 & 3 \end{bmatrix} \quad \Rightarrow p_2 = \frac{1}{2} \text{ trace } \mathbf{A}_2 = -2$$

$$\mathbf{B}_2 = \mathbf{A}_2 + 2\mathbf{I} = \begin{bmatrix} -1 & 4 & 0 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -5 \\ -3 & -3 & -1 & 5 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}\ \mathbf{B}_2 = \begin{bmatrix} -5 & 2 & 0 & -2 \\ 1 & 0 & -2 & -4 \\ -1 & -7 & -3 & 4 \\ 0 & 4 & -2 & -7 \end{bmatrix} \quad \Rightarrow p_3 = \frac{1}{3}\ \text{trace}\ \mathbf{A}_3 = -5$$

$$\mathbf{B}_3 = \mathbf{A}_3 + 5\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 5 & -2 & -4 \\ -1 & -8 & 2 & 4 \\ 0 & 4 & -2 & -2 \end{bmatrix}$$

$$\mathbf{A}_4 = \mathbf{A}\ \mathbf{B}_3 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \Rightarrow \ p_4 = \frac{1}{4}\text{trace}\ \mathbf{A}_4 = -2$$

Thus, characteristic polynomial is $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 5\lambda + 2$

Note that $\mathbf{B}_4 = \mathbf{A}_4 + 2\mathbf{I} = 0$ as required by check.

▪ **6(a)** Eigenvalues given by $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$
so eigenvectors are $\lambda_1 = 2, \lambda_2 = 0$
Eigenvectors given by corresponding solutions of

$$(1 - \lambda_i)e_{i1} + e_{i2} = 0$$
$$e_{i1} + (1 - \lambda_i)e_{i2} = 0$$

Taking $i = 1, 2$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 1]^T, \mathbf{e}_2 = [1\ \ -1]^T \tag{1}$$

6(b) Eigenvalues given by $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$
so eigenvectors are $\lambda_1 = 4, \lambda_2 = -1$
Eigenvectors given by corresponding solutions of

$$(l - \lambda_i)e_{i1} + 2e_{i2} = 0$$
$$3e_{i1} + (2 - \lambda_i)e_{i2} = 0$$

Taking $i = 1, 2$ gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 3]^T, \mathbf{e}_2 = [1\ \ -1]^T$$

6(c) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = \lambda^3 + 9\lambda^2 + 9\lambda - 81 = (\lambda - 9)(\lambda - 3)(\lambda + 3) = 0$$

So the eigenvalues are $\lambda_1 = 9, \lambda_2 = 3, \lambda_3 = -3$.
The eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} + 0e_{i2} - 4e_{i3} = 0$$
$$0e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} = 0$$
$$-4e_{i1} + 4e_{i2} + (3-\lambda_i)e_{i3} = 0$$

Taking $i = 1, \lambda_i = 9$ solution is

$$\frac{e_{11}}{8} = -\frac{e_{12}}{16} = \frac{e_{13}}{-16} = \beta_1 \qquad \Rightarrow \mathbf{e}_1 = [-1\ 2\ 2]^T$$

Taking $i = 2, \lambda_i = 3$ solution is

$$\frac{e_{21}}{-16} = -\frac{e_{22}}{16} = \frac{e_{23}}{8} = \beta_2 \qquad \Rightarrow \mathbf{e}_2 = [2\ 2\ \ -1]^T$$

Taking $i = 3, \lambda_i = -3$ solution is

$$\frac{e_{31}}{32} = -\frac{e_{32}}{16} = \frac{e_{33}}{32} = \beta_3 \qquad \Rightarrow \mathbf{e}_3 = [2\ \ -1\ 2]^T$$

6(d) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

Adding column 1 to column 2 gives

$$\begin{vmatrix} 1-\lambda & 2-\lambda & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix}$$

$$R_1 \underline{=} R_2 (2-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(3-\lambda)$$

so the eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.
Eigenvectors are the corresponding solutions of $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{e}_i = 0$
When $\lambda = \lambda_1 = 3$ we have

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{11}}{-2} = -\frac{e_{12}}{2} = \frac{e_{13}}{-1} = \beta_1$$

so the eigenvector corresponding to $\lambda_2 = 3$ is $\mathbf{e}_1 = \beta_1[2\ 2\ 1]^T, \beta_1$ constant.
When $\lambda = \lambda_2 = 2$ we have

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{21}}{-2} = -\frac{e_{22}}{2} = \frac{e_{23}}{0} = \beta_3$$

so the eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{e}_2 = \beta_2[1\ 1\ 0]^T, \beta_2$ constant.
When $\lambda = \lambda_3 = 1$ we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{31}}{0} = -\frac{e_{32}}{2} = \frac{e_{33}}{1} = \beta_1$$

so the eigenvector corresponding to $\lambda_3 = 1$ is $\mathbf{e}_3 = \beta_3[0\ \ -2\ 1]^T, \beta_3$ constant.

6(e) Eigenvalues given by

$$\begin{vmatrix} 5-\lambda & 0 & 6 \\ 0 & 11-\lambda & 6 \\ 6 & 6 & -2-\lambda \end{vmatrix} = \lambda^3 - 14\lambda^2 - 23\lambda - 686 = (\lambda - 14)(\lambda - 7)(\lambda + 7) = 0$$

so eigenvalues are $\lambda_1 = 14, \lambda_2 = 7, \lambda_3 = -7$

Eigenvectors are given by the corresponding solutions of

$$\begin{aligned}(5-\lambda_i)e_{i1}+0e_{i2}+6e_{i3}&=0\\ 0e_{i1}+(11-\lambda_i)e_{i2}+6e_{i3}&=0\\ 6e_{i1}+6e_{i2}+(-2-\lambda_i)e_{i3}&=0\end{aligned}$$

When $i=1,\lambda_1=14$ solution is

$$\frac{e_{11}}{12}=\frac{-e_{12}}{-36}=\frac{e_{13}}{18}=\beta_1 \;\Rightarrow \mathbf{e}_1=[2\ 6\ 3]^T$$

When $i=2,\lambda_2=7$ solution is

$$\frac{e_{21}}{-72}=\frac{-e_{22}}{-36}=\frac{e_{23}}{-24}=\beta_2 \;\Rightarrow \mathbf{e}_2=[6\ \ -3\ 2]^T$$

When $i=3,\lambda_3=-7$ solution is

$$\frac{e_{31}}{54}=\frac{-e_{32}}{-36}=\frac{e_{33}}{-108}=\beta_3 \;\Rightarrow \mathbf{e}_3=[3\ 2\ \ -6]^T$$

6(f) Eigenvalues given by

$$\begin{vmatrix}1-\lambda & -1 & 0\\ 1 & 2-\lambda & 1\\ -2 & 1 & -1-\lambda\end{vmatrix} \underset{=}{R_1+R_2} \begin{vmatrix}-1-\lambda & 0 & -1-\lambda\\ 1 & 2-\lambda & 1\\ -2 & 1 & -1-\lambda\end{vmatrix}$$

$$=(1+\lambda)\begin{vmatrix}-1 & 0 & 0\\ 1 & 2-\lambda & 0\\ -2 & 1 & 1-\lambda\end{vmatrix}=0,\ \text{i.e. } (1+\lambda)(2-\lambda)(1-\lambda)=0$$

so eigenvalues are $\lambda_1=2,\lambda_2=1,\lambda_3=-1$
Eigenvectors are given by the corresponding solutions of

$$\begin{aligned}(1-\lambda_i)e_{i1}-e_{i2}+0e_{i3}&=0\\ e_{i1}+(2-\lambda_i)e_{i2}+e_{i3}&=0\\ -2e_{i1}+e_{i2}-(1+\lambda_i)e_{i3}&=0\end{aligned}$$

Taking $i=1,2,3$ gives the eigenvectors as

$$\mathbf{e}_1=[-1\ 1\ 1]^T,\mathbf{e}_2=[1\ 0\ \ -1]^T,\mathbf{e}_3=[1\ 2\ \ -7]^T$$

6(g) Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 2 & 5-\lambda & 4 \\ -1 & -1 & -\lambda \end{vmatrix} \underset{=}{R_1 + (R_2 + R_3)} \begin{vmatrix} 5-\lambda & 5-\lambda & 5-\lambda \\ 2 & 5-\lambda & 4 \\ -1 & -1 & -\lambda \end{vmatrix}$$

$$= (5-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(1-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 1$

Eigenvectors are given by the corresponding solutions of

$$(4-\lambda_i)e_{i1} + e_{i2} + e_{i3} = 0$$

$$2e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} = 0$$

$$-e_{i1} - e_{i2} - \lambda_i e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 3\ \ -1]^T, \mathbf{e}_2 = [1\ \ -1\ 0]^T, \mathbf{e}_3 = [0\ \ -1\ 1]^T$$

6(h) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & -4 & -2 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix} \underset{=}{R_1 \pm 2R_2} \begin{vmatrix} 1-\lambda & 2-2\lambda & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(4-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$

Eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} - 4e_{i2} - 2e_{i3} = 0$$

$$2e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$

$$e_{i1} + 2e_{i2} + (4-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2\ \ -1\ \ -1]^T, \mathbf{e}_2 = [2\ \ -1\ \ 0]^T, \mathbf{e}_3 = [4\ \ -1\ \ -2]^T$$

Exercises 1.4.5

- **7(a)** Eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} \underset{=}{R_1 - R_2} \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 3 & 2-\lambda \end{vmatrix} = (1-\lambda)[\lambda^2 - 6\lambda + 5] = (1-\lambda)(\lambda-1)(\lambda-5) = 0$$

so eigenvalues are $\lambda_1 = 5, \lambda_2 = \lambda_3 = 1$

The eigenvectors are the corresponding solutions of

$$(2-\lambda_i)e_{i1} + 2e_{i2} + e_{i3} = 0$$

$$e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$

$$e_{i1} + 2e_{i2} + (2-\lambda_i)e_{i3} = 0$$

When $i = 1, \lambda_1 = 5$ and solution is

$$\frac{e_{11}}{4} = \frac{-e_{12}}{-4} = \frac{e_{13}}{4} = \beta_1 \ \Rightarrow\ \mathbf{e}_1 = [1\ 1\ 1]^T$$

When $\lambda_2 = \lambda_3 = 1$ solution is given by the single equation

$$e_{21} + 2e_{22} + e_{23} = 0$$

Following the procedure of Example 1.6 we can obtain two linearly independent solutions. A possible pair are

$$\mathbf{e}_2 = [0\ 1\ 2]^T\ ,\ \mathbf{e}_3 = [1\ 0\ \ -1]^T$$

7(b) Eigenvalues given by

$$\begin{vmatrix} -\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda+1)(\lambda-2)^2 = 0$$

so eigenvalues are $\lambda_1 = \lambda_2 = 2, \lambda_3 = -1$
The eigenvectors are the corresponding solutions of

$$\begin{aligned} -\lambda_i e_{i1} - 2e_{i2} - 2e_{i3} &= 0 \\ -e_{i1} + (1-\lambda_i)e_{i2} + 2e_{i3} &= 0 \\ -e_{i1} - e_{i2} + (2-\lambda_i)e_{i3} &= 0 \end{aligned}$$

When $i = 3, \lambda_3 = -1$ corresponding solution is

$$\frac{e_{31}}{8} = \frac{-e_{32}}{-1} = \frac{e_{33}}{3} = \beta_3 \;\Rightarrow\; \mathbf{e}_3 = [8\ 1\ 3]^T$$

When $\lambda_1 = \lambda_2 = 2$ solution is given by

$$-2e_{21} - 2e_{22} - 2e_{23} = 0 \qquad (1)$$

$$-e_{21} - e_{22} + 2e_{23} = 0 \qquad (2)$$

$$-e_{21} - e_{22} = 0 \qquad (3)$$

From (1) and (2) $e_{23} = 0$ and it follows from (3) that $e_{21} = -e_{22}$. We deduce that there is only one linearly independent eigenvector corresponding to the repeated eigenvalues $\lambda = 2$. A possible eigenvector is

$$\mathbf{e}_2 = [1\ \ -1\ 0]^T$$

7(c) Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} \underset{=}{R_1 - 3R_3} \begin{vmatrix} 1-\lambda & -3+3\lambda & 0 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & -3 & 0 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 6-\lambda & 2 \\ 1 & -8 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 + \lambda + 4) = (1-\lambda)(\lambda-2)^2 = 0$$

so eigenvalues are $\lambda_1 = \lambda_2 = 2,\ \lambda_3 = 1$.
The eigenvectors are the corresponding solutions of

$$(4 - \lambda_i)e_{i1} + 6e_{i2} + 6e_{i3} = 0$$
$$e_{i1} + (3 - \lambda_i)e_{i2} + 2e_{i3} = 0$$
$$-e_{i1} - 5e_{i2} - (2 + \lambda_i)e_{i3} = 0$$

When $i = 3, \lambda_3 = 1$ corresponding solution is

$$\frac{e_{31}}{4} = \frac{-e_{32}}{-1} = \frac{e_{33}}{-3} = \beta_3 \ \Rightarrow\ \mathbf{e}_3 = [4\ 1\ \ -3]^T$$

When $\lambda_1 = \lambda_2 = 2$ solution is given by

$$2e_{21} + 6e_{22} + 6e_{23} = 0$$
$$e_{21} + e_{22} + 2e_{23} = 0$$
$$-e_{21} - 5e_{22} - 4e_{23} = 0$$

so that

$$\frac{e_{21}}{6} = \frac{-e_{22}}{-2} = \frac{e_{23}}{-4} = \beta_2$$

leading to only one linearly eigenvector corresponding to the eigenvector $\lambda = 2$. A possible eigenvector is

$$\mathbf{e}_2 = [3\ 1\ \ -2]^T$$

7(d) Eigenvalues given by

$$\begin{vmatrix} 7-\lambda & -2 & -4 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix} \underset{=}{R_1 - 2R_2} \begin{vmatrix} 1-\lambda & -2+2\lambda & 0 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix}$$
$$= (1-\lambda)\begin{vmatrix} 1 & -2 & 0 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 6-\lambda & -2 \\ 6 & 10 & -3-\lambda \end{vmatrix}$$
$$= (1-\lambda)(\lambda-2)(\lambda-1) = 0$$

so eigenvalues are $\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 1$.
The eigenvectors are the corresponding solutions of

$$(7 - \lambda_i)e_{i1} - 2e_{i2} - 4e_{i3} = 0$$
$$3e_{i1} - \lambda_i e_{i2} - 2e_{i3} = 0$$
$$6e_{i1} - 2e_{i2} - (3 + \lambda_i)e_{i3} = 0$$

When $i = 1, \lambda_2 = 2$ and solution is

$$\frac{e_{11}}{6} = \frac{-e_{12}}{-3} = \frac{e_{13}}{6} = \beta_1 \Rightarrow \mathbf{e}_1 = [2\ 1\ 2]^T$$

When $\lambda_2 = \lambda_3 = 1$ the solution is given by the single equation

$$3e_{21} - e_{22} - 2e_{23} = 0$$

Following the procedures of Example 1.6 we can obtain two linearly independent solutions. A possible pair are

$$\mathbf{e}_2 = [0\ 2\ \ -1]^T\ , \mathbf{e}_3 = [2\ 0\ 3]^T$$

■ **8**

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Performing a series of row and column operators this may be reduced to the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ indicating that $(\mathbf{A} - \mathbf{I})$ is of rank 2. Thus, the nullity $q = 3 - 2 = 1$ confirming that there is only one linearly independent eigenvector associated with the eigenvalue $\lambda = 1$. The eigenvector is given by the solution of

$$\begin{aligned} -4e_{11} - 7e_{12} - 5e_{13} &= 0 \\ 2e_{11} + 3e_{12} + 3e_{13} &= 0 \\ e_{11} + 2e_{12} + e_{13} &= 0 \end{aligned}$$

giving

$$\frac{e_{11}}{-3} = \frac{-e_{12}}{-1} = \frac{e_{13}}{1} = \beta_1 \Rightarrow \mathbf{e}_1 = [-3\ 1\ 1]^T$$

■ **9**

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Performing a series of row and column operators this may be reduced to the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ indicating that $(\mathbf{A}-\mathbf{I})$ is of rank 1. Then, the nullity of $q = 3-1 = 2$ confirming that there are two linearly independent eigenvectors associated with the eigenvalue $\lambda = 1$. The eigenvectors are given by the single equation

$$e_{11} + e_{12} - e_{13} = 0$$

and two possible linearly independent eigenvectors are

$$\mathbf{e}_1 = [1\ 0\ 1]^T \text{ and } \mathbf{e}_2 = [0\ 1\ 1]^T$$

Exercises 1.4.8

- **10** These are standard results.

- **11(a)** (i) Trace $\mathbf{A} = 4 + 5 + 0 = 9 =$ sum eigenvalues;

  (ii) $\det \mathbf{A} = 15 = 5 \times 3 \times 1 =$ product eigenvalues;

  (iii) $\mathbf{A}^{-1} = \dfrac{1}{15}\begin{bmatrix} 4 & -1 & -1 \\ -4 & 1 & -14 \\ 3 & 3 & 18 \end{bmatrix}$. Eigenvalues given by

$$\begin{vmatrix} 4-15\lambda & -1 & -1 \\ -4 & 1-15\lambda & -14 \\ 3 & 3 & 18-15\lambda \end{vmatrix} \underset{=}{C_3 - C_2} \begin{vmatrix} 4-15\lambda & -1 & 0 \\ -4 & 1-15\lambda & -15+15\lambda \\ 3 & 3 & 15-15\lambda \end{vmatrix}$$

$$= (15-15\lambda)\begin{vmatrix} 4-15\lambda & -1 & 0 \\ -4 & 1-15\lambda & -1 \\ 3 & 3 & 1 \end{vmatrix} = (15-15\lambda)(15\lambda-5)(15\lambda-3) = 0$$

confirming eigenvalues as $1, \frac{1}{3}, \frac{1}{5}$.

(iv) $A^T = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 5 & -1 \\ 1 & 4 & 0 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 2 & -1 \\ 1 & 5-\lambda & -1 \\ 1 & 4 & -\lambda \end{vmatrix} = (\lambda-5)(\lambda-3)(\lambda-1) = 0$$

that is, eigenvalue as for $\mathbf{A}$.

11(b) (i) $2\mathbf{A} = \begin{bmatrix} 8 & 2 & 2 \\ 4 & 10 & 8 \\ -2 & -2 & 0 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 8-\lambda & 2 & 2 \\ 4 & 10-\lambda & 8 \\ -2 & -2 & -\lambda \end{vmatrix} \underset{=}{C_1 - C_2} \begin{vmatrix} 6-\lambda & 2 & 2 \\ -6+\lambda & 10-\lambda & 8 \\ 0 & -2 & -\lambda \end{vmatrix}$$
$$= (6-\lambda)\begin{vmatrix} 1 & 2 & 2 \\ -1 & 10-\lambda & 8 \\ 0 & -2 & -\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 1 & 2 & 2 \\ 0 & 12-\lambda & 10 \\ 0 & -2 & -\lambda \end{vmatrix}$$
$$= (6-\lambda)(\lambda-10)(\lambda-2) = 0$$

Thus eigenvalues are 2 times those of $\mathbf{A}$; namely 6, 10 and 2.

(ii) $\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 7 & 4 \\ -1 & -1 & 2 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 6-\lambda & 1 & 1 \\ 2 & 7-\lambda & 4 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 15\lambda^2 - 71\lambda + 105 = -(\lambda-7)(\lambda-5)(\lambda-3) = 0$$

confirming the eigenvalues as $5+2, 3+2, 1+2$.

Likewise for $\mathbf{A} - 2\mathbf{I}$