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CHAPTER 1

PRELIMINARIES

We suggest that this chapter be treated as review and covered quickly, without detailed classroom discussion. For one reason, many of these ideas will be already familiar to the students — at least informally. Further, we believe that, in practice, those notions of importance are best learned in the arena of real analysis, where their use and significance are more apparent. Dwelling on the formal aspect of sets and functions does not contribute very greatly to the students' understanding of real analysis.

If the students have already studied abstract algebra, number theory or combinatorics, they should be familiar with the use of mathematical induction. If not, then some time should be spent on mathematical induction.

The third section deals with finite, infinite and countable sets. These notions are important and should be briefly introduced. However, we believe that it is not necessary to go into the proofs of these results at this time.

Section 1.1

Students are usually familiar with the notations and operations of set algebra, so that a brief review is quite adequate. One item that should be mentioned is that two sets A and B are often proved to be equal by showing that: (i) if $x \in A$, then $x \in B$, and (ii) if $x \in B$, then $x \in A$. This type of element-wise argument is very common in real analysis, since manipulations with set identities is often not suitable when the sets are complicated.

Students are often not familiar with the notions of functions that are injective (= one-one) or surjective (= onto).

Sample Assignment: Exercises 1, 3, 9, 14, 15, 20.

Partial Solutions:

1. (a) $B \cap C = \{5, 11, 17, 23, \ldots\} = \{6k - 1 : k \in \mathbb{N}\}$, $A \cap (B \cap C) = \{5, 11, 17\}$
   (b) $(A \cap B) \setminus C = \{2, 8, 14, 20\}$
   (c) $(A \cap C) \setminus B = \{3, 7, 9, 13, 15, 19\}$
2. The sets are equal to (a) A, (b) $A \cap B$, (c) the empty set.
3. If $A \subseteq B$, then $x \in A$ implies $x \in B$, whence $x \in A \cap B$, so that $A \subseteq A \cap B \subseteq A$. Thus, if $A \subseteq B$, then $A = A \cap B$.
   Conversely, if $A = A \cap B$, then $x \in A$ implies $x \in A \cap B$, whence $x \in B$. Thus if $A = A \cap B$, then $A \subseteq B$.
4. If x is in $A \setminus (B \cap C)$, then x is in A but $x \notin B \cap C$, so that $x \in A$ and x is either not in B or not in C. Therefore either $x \in A \setminus B$ or $x \in A \setminus C$, which implies that $x \in (A \setminus B) \cup (A \setminus C)$. Thus $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Conversely, if x is in $(A\setminus B)\cup(A\setminus C)$, then $x\in A\setminus B$ or $x\in A\setminus C$. Thus $x\in A$ and either $x\notin B$ or $x\notin C$, which implies that $x\in A$ but $x\notin B\cap C$, so that $x\in A\setminus(B\cap C)$. Thus $(A\setminus B)\cup(A\setminus C)\subseteq A\setminus(B\cap C)$.

Since the sets $A\setminus(B\cap C)$ and $(A\setminus B)\cup(A\setminus C)$ contain the same elements, they are equal.

5. (a) If $x\in A\cap(B\cup C)$, then $x\in A$ and $x\in B\cup C$. Hence we either have (i) $x\in A$ and $x\in B$, or we have (ii) $x\in A$ and $x\in C$. Therefore, either $x\in A\cap B$ or $x\in A\cap C$, so that $x\in(A\cap B)\cup(A\cap C)$. This shows that $A\cap(B\cup C)$ is a subset of $(A\cap B)\cup(A\cap C)$.

   Conversely, let y be an element of $(A\cap B)\cup(A\cap C)$. Then either (j) $y\in A\cap B$, or (jj) $y\in A\cap C$. It follows that $y\in A$ and either $y\in B$ or $y\in C$. Therefore, $y\in A$ and $y\in B\cup C$, so that $y\in A\cap(B\cup C)$. Hence $(A\cap B)\cup(A\cap C)$ is a subset of $A\cap(B\cup C)$.

   In view of Definition 1.1.1, we conclude that the sets $A\cap(B\cup C)$ and $(A\cap B)\cup(A\cap C)$ are equal.
   (b) Similar to (a).
6. The set D is the union of $\{x : x\in A \text{ and } x\notin B\}$ and $\{x : x\notin A \text{ and } x\in B\}$.
7. Here $A_n=\{n+1, 2(n+1), \ldots\}$.
   (a) $A_1=\{2,4,6,8,\ldots\}$, $A_2=\{3,6,9,12,\ldots\}$, $A_1\cap A_2=\{6,12,18,24,\ldots\}=\{6k : k\in\mathbb{N}\}=A_5$.
   (b) $\bigcup A_n=\mathbb{N}\setminus\{1\}$, because if $n>1$, then $n\in A_{n-1}$; moreover $1\notin A_n$. Also $\bigcap A_n=\emptyset$, because $n\notin A_n$ for any $n\in\mathbb{N}$.
8. (a) The graph consists of four horizontal line segments.
   (b) The graph consists of three vertical line segments.
9. No. For example, both (0, 1) and (0, − 1) belong to C.
10. (a) $f(E)=\{1/x^2 : 1\le x\le 2\}=\{y : \frac{1}{4}\le y\le 1\}=[\frac{1}{4},1]$.
    (b) $f^{-1}(G)=\{x : 1\le 1/x^2\le 4\}=\{x : \frac{1}{4}\le x^2\le 1\}=[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]$.
11. (a) $f(E)=\{x+2 : 0\le x\le 1\}=[2,3]$, so $h(E)=g(f(E))=g([2,3])=\{y^2 : 2\le y\le 3\}=[4,9]$.
    (b) $g^{-1}(G)=\{y : 0\le y^2\le 4\}=[-2,2]$, so $h^{-1}(G)=f^{-1}(g^{-1}(G))=f^{-1}([-2,2])=\{x : -2\le x+2\le 2\}=[-4,0]$.
12. If 0 is removed from E and F, then their intersection is empty, but the intersection of the images under f is $\{y : 0<y\le 1\}$.
13. $E\setminus F=\{x : -1\le x<0\}$, $f(E)\setminus f(F)$ is empty, and $f(E\setminus F)=\{y : 0<y\le 1\}$.
14. If $y\in f(E\cap F)$, then there exists $x\in E\cap F$ such that $y=f(x)$. Since $x\in E$ implies $y\in f(E)$, and $x\in F$ implies $y\in f(F)$, we have $y\in f(E)\cap f(F)$. This proves $f(E\cap F)\subseteq f(E)\cap f(F)$.
15. If $x\in f^{-1}(G)\cap f^{-1}(H)$, then $x\in f^{-1}(G)$ and $x\in f^{-1}(H)$, so that $f(x)\in G$ and $f(x)\in H$. Then $f(x)\in G\cap H$, and hence $x\in f^{-1}(G\cap H)$. This shows

that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$. The opposite inclusion is shown in Example 1.1.8(b). The proof for unions is similar.

16. If $f(a) = f(b)$, then $a/\sqrt{a^2+1} = b/\sqrt{b^2+1}$, from which it follows that $a^2 = b^2$. Since a and b must have the same sign, we get $a = b$, and hence f is injective. If $-1 < y < 1$, then $x := y/\sqrt{1-y^2}$ satisfies $f(x) = y$ (why?), so that f takes $\mathbb{R}$ onto the set $\{y : -1 < y < 1\}$. If $x > 0$, then $x = \sqrt{x^2} < \sqrt{x^2+1}$, so it follows that $f(x) \in \{y : 0 < y < 1\}$.
17. One bijection is the familiar linear function that maps a to 0 and b to 1, namely, $f(x) := (x-a)/(b-a)$. Show that this function works.
18. (a) Let $f(x) = 2x$, $g(x) = 3x$.
    (b) Let $f(x) = x^2$, $g(x) = x$, $h(x) = 1$. (Many examples are possible.)
19. (a) If $x \in f^{-1}(f(E))$, then $f(x) \in f(E)$, so that there exists $x_1 \in E$ such that $f(x_1) = f(x)$. If f is injective, then $x_1 = x$, whence $x \in E$. Therefore, $f^{-1}(f(E)) \subseteq E$. Since $E \subseteq f^{-1}(f(E))$ holds for any f, we have set equality when f is injective. See Example 1.1.8(a) for an example.
    (b) If $y \in H$ and f is surjective, then there exists $x \in A$ such that $f(x) = y$. Then $x \in f^{-1}(H)$ so that $y \in f(f^{-1}(H))$. Therefore $H \subseteq f(f^{-1}(H))$. Since $f(f^{-1}(H)) \subseteq H$ for any f, we have set equality when f is surjective. See Example 1.1.8(a) for an example.
20. (a) Since $y = f(x)$ if and only if $x = f^{-1}(y)$, it follows that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.
    (b) Since f is injective, then f^{-1} is injective on $R(f)$. And since f is surjective, then f^{-1} is defined on $R(f) = B$.
21. If $g(f(x_1)) = g(f(x_2))$, then $f(x_1) = f(x_2)$, so that $x_1 = x_2$, which implies that $g \circ f$ is injective. If $w \in C$, there exists $y \in B$ such that $g(y) = w$, and there exists $x \in A$ such that $f(x) = y$. Then $g(f(x)) = w$, so that $g \circ f$ is surjective. Thus $g \circ f$ is a bijection.
22. (a) If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, which implies $x_1 = x_2$, since $g \circ f$ is injective. Thus f is injective.
    (b) Given $w \in C$, since $g \circ f$ is surjective, there exists $x \in A$ such that $g(f(x)) = w$. If $y := f(x)$, then $y \in B$ and $g(y) = w$. Thus g is surjective.
23. We have $x \in f^{-1}(g^{-1}(H)) \iff f(x) \in g^{-1}(H) \iff g(f(x)) \in H \iff x \in (g \circ f)^{-1}(H)$.
24. If $g(f(x)) = x$ for all $x \in D(f)$, then $g \circ f$ is injective, and Exercise 22(a) implies that f is injective on $D(f)$. If $f(g(y)) = y$ for all $y \in D(g)$, then Exercise 22(b) implies that f maps $D(f)$ onto $D(g)$. Thus f is a bijection of $D(f)$ onto $D(g)$, and $g = f^{-1}$.

Section 1.2

The method of proof known as Mathematical Induction is used frequently in real analysis, but in many situations the details follow a routine patterns and are left to the reader by means of a phrase such as: "The proof is by Mathematical Induction". Since may students have only a hazy idea of what is involved, it may be a good idea to spend some time explaining and illustrating what constitutes a proof by induction.

Pains should be taken to emphasize that the induction hypothesis does *not* entail "assuming what is to be proved". The inductive step concerns the validity of going from the assertion for $k \in \mathbb{N}$ to that for $k+1$. The truth of falsity of the individual assertion is not an issue here.

Sample Assignment: Exercises 1, 2, 6, 11, 13, 14, 20.

Partial Solutions:

1. The assertion is true for $n=1$ because $1/(1\cdot 2)=1/(1+1)$. If it is true for $n=k$, then it follows for $k+1$ because $k/(k+1)+1/[(k+1)(k+2)]=(k+1)/(k+2)$.
2. The statement is true for $n=1$ because $[\frac{1}{2}\cdot 1\cdot 2]^2=1=1^3$. For the inductive step, use the fact that
$$\left[\tfrac{1}{2}k(k+1)\right]^2+(k+1)^3=\left[\tfrac{1}{2}(k+1)(k+2)\right]^2.$$
3. It is true for $n=1$ since $3=4-1$. If the equality holds for $n=k$, then add $8(k+1)-5=8k+3$ to both sides and show that $(4k^2-k)+(8k+3)=4(k+1)^2-(k+1)$ to deduce equality for the case $n=k+1$.
4. It is true for $n=1$ since $1=(4-1)/3$. If it is true for $n=k$, then add $(2k+1)^2$ to both sides and use some algebra to show that
$$\tfrac{1}{3}(4k^3-k)+(2k+1)^2=\tfrac{1}{3}[4k^3+12k^2+11k+3]=\tfrac{1}{3}[4(k+1)^3-(k+1)],$$
which establishes the case $n=k+1$.
5. Equality holds for $n=1$ since $1^2=(-1)^2(1\cdot 2)/2$. The proof is completed by showing $(-1)^{k+1}[k(k+1)]/2+(-1)^{k+2}(k+1)^2=(-1)^{k+2}[(k+1)(k+2)]/2$.
6. If $n=1$, then $1^3+5\cdot 1=6$ is divisible by 6. If k^3+5k is divisible by 6, then $(k+1)^3+5(k+1)=(k^3+5k)+3k(k+1)+6$ is also, because $k(k+1)$ is always even (why?) so that $3k(k+1)$ is divisible by 6, and hence the sum is divisible by 6.
7. If $5^{2k}-1$ is divisible by 8, then it follows that $5^{2(k+1)}-1=(5^{2k}-1)+24\cdot 5^{2k}$ is also divisible by 8.
8. $5^{k+1}-4(k+1)-1=5\cdot 5^k-4k-5=(5^k-4k-1)+4(5^k-1)$. Now show that 5^k-1 is always divisible by 4.
9. If $k^3+(k+1)^3+(k+2)^3$ is divisible by 9, then $(k+1)^3+(k+2)^3+(k+3)^3=k^3+(k+1)^3+(k+2)^3+9(k^2+3k+3)$ is also divisible by 9.
10. The sum is equal to $n/(2n+1)$.