CHAPTER 1

Solution to Problem 1.1. We have

$$A = \{2, 4, 6\}, \qquad B = \{4, 5, 6\},$$

so $A \cup B = \{2, 4, 5, 6\}$, and

$$(A \cup B)^c = \{1, 3\}.$$

On the other hand,

$$A^c \cap B^c = \{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}.$$

Similarly, we have $A \cap B = \{4, 6\}$, and

$$(A \cap B)^c = \{1, 2, 3, 5\}.$$

On the other hand,

$$A^c \cup B^c = \{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}.$$

Solution to Problem 1.2. (a) By using a Venn diagram it can be seen that for any sets S and T, we have

$$S = (S \cap T) \cup (S \cap T^c).$$

(Alternatively, argue that any x must belong to either T or to T^c, so x belongs to S if and only if it belongs to $S \cap T$ or to $S \cap T^c$.) Apply this equality with $S = A^c$ and $T = B$, to obtain the first relation

$$A^c = (A^c \cap B) \cup (A^c \cap B^c).$$

Interchange the roles of A and B to obtain the second relation.

(b) By De Morgan's law, we have

$$(A \cap B)^c = A^c \cup B^c,$$

and by using the equalities of part (a), we obtain

$$(A \cap B)^c = \big((A^c \cap B) \cup (A^c \cap B^c)\big) \cup \big((A \cap B^c) \cup (A^c \cap B^c)\big) = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c).$$

(c) We have $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, so $A \cap B = \{1, 3\}$. Therefore,

$$(A \cap B)^c = \{2, 4, 5, 6\},$$

and

$$A^c \cap B = \{2\}, \qquad A^c \cap B^c = \{4,6\}, \qquad A \cap B^c = \{5\}.$$

Thus, the equality of part (b) is verified.

Solution to Problem 1.5. Let G and C be the events that the chosen student is a genius and a chocolate lover, respectively. We have $\mathbf{P}(G) = 0.6$, $\mathbf{P}(C) = 0.7$, and $\mathbf{P}(G \cap C) = 0.4$. We are interested in $\mathbf{P}(G^c \cap C^c)$, which is obtained with the following calculation:

$$\mathbf{P}(G^c \cap C^c) = 1 - \mathbf{P}(G \cup C) = 1 - \big(\mathbf{P}(G) + \mathbf{P}(C) - \mathbf{P}(G \cap C)\big) = 1 - (0.6 + 0.7 - 0.4) = 0.1.$$

Solution to Problem 1.6. We first determine the probabilities of the six possible outcomes. Let $a = \mathbf{P}(\{1\}) = \mathbf{P}(\{3\}) = \mathbf{P}(\{5\})$ and $b = \mathbf{P}(\{2\}) = \mathbf{P}(\{4\}) = \mathbf{P}(\{6\})$. We are given that $b = 2a$. By the additivity and normalization axioms, $1 = 3a + 3b = 3a + 6a = 9a$. Thus, $a = 1/9$, $b = 2/9$, and $\mathbf{P}(\{1,2,3\}) = 4/9$.

Solution to Problem 1.7. The outcome of this experiment can be any finite sequence of the form $(a_1, a_2, \ldots, a_n)$, where n is an arbitrary positive integer, $a_1, a_2, \ldots, a_{n-1}$ belong to $\{1,3\}$, and a_n belongs to $\{2,4\}$. In addition, there are possible outcomes in which an even number is never obtained. Such outcomes are infinite sequences $(a_1, a_2, \ldots)$, with each element in the sequence belonging to $\{1,3\}$. The sample space consists of all possible outcomes of the above two types.

Solution to Problem 1.8. Let p_i be the probability of winning against the opponent played in the ith turn. Then, you will win the tournament if you win against the 2nd player (probability p_2) and also you win against at least one of the two other players [probability $p_1 + (1 - p_1)p_3 = p_1 + p_3 - p_1 p_3$]. Thus, the probability of winning the tournament is

$$p_2(p_1 + p_3 - p_1 p_3).$$

The order $(1,2,3)$ is optimal if and only if the above probability is no less than the probabilities corresponding to the two alternative orders, i.e.,

$$p_2(p_1 + p_3 - p_1 p_3) \geq p_1(p_2 + p_3 - p_2 p_3),$$

$$p_2(p_1 + p_3 - p_1 p_3) \geq p_3(p_2 + p_1 - p_2 p_1).$$

It can be seen that the first inequality above is equivalent to $p_2 \geq p_1$, while the second inequality above is equivalent to $p_2 \geq p_3$.

Solution to Problem 1.9. (a) Since $\Omega = \cup_{i=1}^n S_i$, we have

$$A = \bigcup_{i=1}^{n} (A \cap S_i),$$

while the sets $A \cap S_i$ are disjoint. The result follows by using the additivity axiom.

(b) The events $B \cap C^c$, $B^c \cap C$, $B \cap C$, and $B^c \cap C^c$ form a partition of Ω, so by part (a), we have

$$\mathbf{P}(A) = \mathbf{P}(A \cap B \cap C^c) + \mathbf{P}(A \cap B^c \cap C) + \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C^c). \quad (1)$$

The event $A \cap B$ can be written as the union of two disjoint events as follows:

$$A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C^c),$$

so that

$$\mathbf{P}(A \cap B) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B \cap C^c). \tag{2}$$

Similarly,

$$\mathbf{P}(A \cap C) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C). \tag{3}$$

Combining Eqs. (1)-(3), we obtain the desired result.

Solution to Problem 1.10. Since the events $A \cap B^c$ and $A^c \cap B$ are disjoint, we have using the additivity axiom repeatedly,

$$\mathbf{P}\big((A\cap B^c)\cup(A^c\cap B)\big) = \mathbf{P}(A\cap B^c)+\mathbf{P}(A^c\cap B) = \mathbf{P}(A)-\mathbf{P}(A\cap B)+\mathbf{P}(B)-\mathbf{P}(A\cap B).$$

Solution to Problem 1.14. (a) Each possible outcome has probability $1/36$. There are 6 possible outcomes that are doubles, so the probability of doubles is $6/36 = 1/6$.

(b) The conditioning event (sum is 4 or less) consists of the 6 outcomes

$$\big\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\big\},$$

2 of which are doubles, so the conditional probability of doubles is $2/6 = 1/3$.

(c) There are 11 possible outcomes with at least one 6, namely, $(6,6)$, $(6,i)$, and $(i,6)$, for $i = 1,2,\ldots,5$. Thus, the probability that at least one die is a 6 is $11/36$.

(d) There are 30 possible outcomes where the dice land on different numbers. Out of these, there are 10 outcomes in which at least one of the rolls is a 6. Thus, the desired conditional probability is $10/30 = 1/3$.

Solution to Problem 1.15. Let A be the event that the first toss is a head and let B be the event that the second toss is a head. We must compare the conditional probabilities $\mathbf{P}(A \cap B \,|\, A)$ and $\mathbf{P}(A \cap B \,|\, A \cup B)$. We have

$$\mathbf{P}(A \cap B \,|\, A) = \frac{\mathbf{P}\big((A \cap B) \cap A\big)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)},$$

and

$$\mathbf{P}(A \cap B \,|\, A \cup B) = \frac{\mathbf{P}\big((A \cap B) \cap (A \cup B)\big)}{\mathbf{P}(A \cup B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)}.$$

Since $\mathbf{P}(A \cup B) \geq \mathbf{P}(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes HH, HT, TH, TT are equally likely, we have

$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \qquad \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

A generalization of Alice's reasoning is that if A', B', and C' are events such that $B' \subset C'$ and $A' \cap B' = A' \cap C'$ (for example if $A' \subset B' \subset C'$), then the event

A' is at least as likely if we know that B' has occurred than if we know that C' has occurred. Alice's reasoning corresponds to the special case where $A' = A \cup B$, $B' = A$, and $C' = A \cup B$.

Solution to Problem 1.16. In this problem, there is a tendency to reason that since the opposite face is either heads or tails, the desired probability is 1/2. This is, however, wrong, because given that heads came up, it is more likely that the two-headed coin was chosen. The correct reasoning is to calculate the conditional probability

$$\begin{aligned} p &= \mathbf{P}(\text{two-headed coin was chosen} \,|\, \text{heads came up}) \\ &= \frac{\mathbf{P}(\text{two-headed coin was chosen and heads came up})}{\mathbf{P}(\text{heads came up})}. \end{aligned}$$

We have

$$\mathbf{P}(\text{two-headed coin was chosen and heads came up}) = \frac{1}{3},$$

$$\mathbf{P}(\text{heads came up}) = \frac{1}{2},$$

so by taking the ratio of the above two probabilities, we obtain $p = 2/3$. Thus, the probability that the opposite face is tails is $1 - p = 1/3$.

Solution to Problem 1.17. Let A be the event that the batch will be accepted. Then $A = A_1 \cap A_2 \cap A_3 \cap A_4$, where A_i, $i = 1, \ldots, 4$, is the event that the ith item is not defective. Using the multiplication rule, we have

$$\mathbf{P}(A) = \mathbf{P}(A_1)\mathbf{P}(A_2 \,|\, A_1)\mathbf{P}(A_3 \,|\, A_1 \cap A_2)\mathbf{P}(A_4 \,|\, A_1 \cap A_2 \cap A_3) = \frac{95}{100} \cdot \frac{94}{99} \cdot \frac{93}{98} \cdot \frac{92}{97} = 0.812.$$

Solution to Problem 1.18. Using the definition of conditional probabilities, we have

$$\mathbf{P}(A \cap B \,|\, B) = \frac{\mathbf{P}(A \cap B \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \mathbf{P}(A \,|\, B).$$

Solution to Problem 1.19. Let A be the event that Alice does not find her paper in drawer i. Since the paper is in drawer i with probability p_i, and her search is successful with probability d_i, the multiplication rule yields $\mathbf{P}(A^c) = p_i d_i$, so that $\mathbf{P}(A) = 1 - p_i d_i$. Let B be the event that the paper is in drawer j. If $j \neq i$, then $A \cap B = B$, $\mathbf{P}(A \cap B) = \mathbf{P}(B)$, and we have

$$\mathbf{P}(B \,|\, A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)}{\mathbf{P}(A)} = \frac{p_j}{1 - p_i d_i}.$$

Similarly, if $i = j$, we have

$$\mathbf{P}(B \,|\, A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)\mathbf{P}(A \,|\, B)}{\mathbf{P}(A)} = \frac{p_i(1 - d_i)}{1 - p_i d_i}.$$

Solution to Problem 1.20. (a) Figure 1.1 provides a sequential description for the three different strategies. Here we assume 1 point for a win, 0 for a loss, and 1/2 point

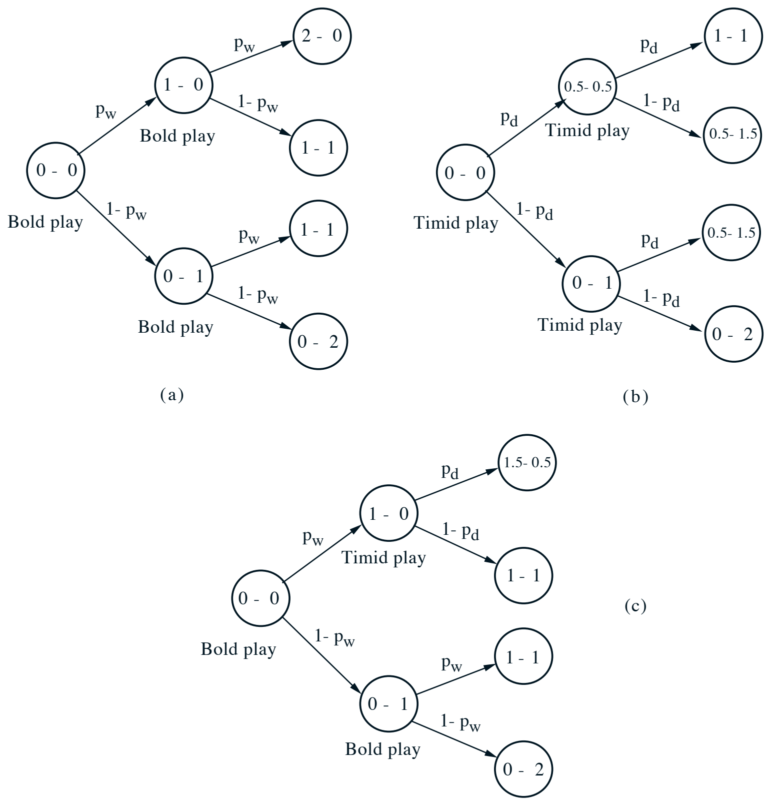

Figure 1.1: Sequential descriptions of the chess match histories under strategies (i), (ii), and (iii).

for a draw. In the case of a tied 1-1 score, we go to sudden death in the next game, and Boris wins the match (probability p_w), or loses the match (probability $1 - p_w$).

(i) Using the total probability theorem and the sequential description of Fig. 1.1(a), we have

$$\mathbf{P}(\text{Boris wins}) = p_w^2 + 2p_w(1 - p_w)p_w.$$

The term p_w^2 corresponds to the win-win outcome, and the term $2p_w(1 - p_w)p_w$ corresponds to the win-lose-win and the lose-win-win outcomes.

(ii) Using Fig. 1.1(b), we have

$$\mathbf{P}(\text{Boris wins}) = p_d^2 p_w,$$

corresponding to the draw-draw-win outcome.

(iii) Using Fig. 1.1(c), we have

$$\mathbf{P}(\text{Boris wins}) = p_w p_d + p_w(1 - p_d)p_w + (1 - p_w)p_w^2.$$

The term $p_w p_d$ corresponds to the win-draw outcome, the term $p_w(1-p_d)p_w$ corresponds to the win-lose-win outcome, and the term $(1-p_w)p_w^2$ corresponds to lose-win-win outcome.

(b) If $p_w < 1/2$, Boris has a greater probability of losing rather than winning any one game, regardless of the type of play he uses. Despite this, the probability of winning the match with strategy (iii) can be greater than 1/2, provided that p_w is close enough to 1/2 and p_d is close enough to 1. As an example, if $p_w = 0.45$ and $p_d = 0.9$, with strategy (iii) we have

$$\mathbf{P}(\text{Boris wins}) = 0.45 \cdot 0.9 + 0.45^2 \cdot (1-0.9) + (1-0.45) \cdot 0.45^2 \approx 0.54.$$

With strategies (i) and (ii), the corresponding probabilities of a win can be calculated to be approximately 0.43 and 0.36, respectively. What is happening here is that with strategy (iii), Boris is allowed to select a playing style *after* seeing the result of the first game, while his opponent is not. Thus, by being able to dictate the playing style in each game after receiving partial information about the match's outcome, Boris gains an advantage.

Solution to Problem 1.21. Let $p(m,k)$ be the probability that the starting player wins when the jar initially contains m white and k black balls. We have, using the total probability theorem,

$$p(m,k) = \frac{m}{m+k} + \frac{k}{m+k}\big(1-p(m,k-1)\big) = 1 - \frac{k}{m+k}p(m,k-1).$$

The probabilities $p(m,1), p(m,2), \ldots, p(m,n)$ can be calculated sequentially using this formula, starting with the initial condition $p(m,0) = 1$.

Solution to Problem 1.22. We derive a recursion for the probability p_i that a white ball is chosen from the ith jar. We have, using the total probability theorem,

$$p_{i+1} = \frac{m+1}{m+n+1}p_i + \frac{m}{m+n+1}(1-p_i) = \frac{1}{m+n+1}p_i + \frac{m}{m+n+1},$$

starting with the initial condition $p_1 = m/(m+n)$. Thus, we have

$$p_2 = \frac{1}{m+n+1} \cdot \frac{m}{m+n} + \frac{m}{m+n+1} = \frac{m}{m+n}.$$

More generally, this calculation shows that if $p_{i-1} = m/(m+n)$, then $p_i = m/(m+n)$. Thus, we obtain $p_i = m/(m+n)$ for all i.

Solution to Problem 1.23. Let $p_{i,n-i}(k)$ denote the probability that after k exchanges, a jar will contain i balls that started in that jar and $n-i$ balls that started in the other jar. We want to find $p_{n,0}(4)$. We argue recursively, using the total probability theorem. We have

$$\begin{aligned} p_{n,0}(4) &= \frac{1}{n}\cdot\frac{1}{n}\cdot p_{n-1,1}(3),\\ p_{n-1,1}(3) &= p_{n,0}(2) + 2\cdot\frac{n-1}{n}\cdot\frac{1}{n}\cdot p_{n-1,1}(2) + \frac{2}{n}\cdot\frac{2}{n}\cdot p_{n-2,2}(2),\\ p_{n,0}(2) &= \frac{1}{n}\cdot\frac{1}{n}\cdot p_{n-1,1}(1),\\ p_{n-1,1}(2) &= 2\cdot\frac{n-1}{n}\cdot\frac{1}{n}\cdot p_{n-1,1}(1),\\ p_{n-2,2}(2) &= \frac{n-1}{n}\cdot\frac{n-1}{n}\cdot p_{n-1,1}(1),\\ p_{n-1,1}(1) &= 1. \end{aligned}$$

Combining these equations, we obtain

$$p_{n,0}(4) = \frac{1}{n^2}\left(\frac{1}{n^2} + \frac{4(n-1)^2}{n^4} + \frac{4(n-1)^2}{n^4}\right) = \frac{1}{n^2}\left(\frac{1}{n^2} + \frac{8(n-1)^2}{n^4}\right).$$

Solution to Problem 1.24. Intuitively, there is something wrong with this rationale. The reason is that it is not based on a correctly specified probabilistic model. In particular, the event where both of the other prisoners are to be released is not properly accounted in the calculation of the posterior probability of release.

To be precise, let A, B, and C be the prisoners, and let A be the one who considers asking the guard. Suppose that all prisoners are a priori equally likely to be released. Suppose also that if B and C are to be released, then the guard chooses B or C with equal probability to reveal to A. Then, there are four possible outcomes:

(1) A and B are to be released, and the guard says B (probability 1/3).

(2) A and C are to be released, and the guard says C (probability 1/3).

(3) B and C are to be released, and the guard says B (probability 1/6).

(4) B and C are to be released, and the guard says C (probability 1/6).

Thus,

$$\begin{aligned} \mathbf{P}(\text{A is to be released} \mid \text{guard says B}) &= \frac{\mathbf{P}(\text{A is to be released and guard says B})}{\mathbf{P}(\text{guard says B})}\\ &= \frac{1/3}{1/3+1/6} = \frac{2}{3}. \end{aligned}$$

Similarly,

$$\mathbf{P}(\text{A is to be released} \mid \text{guard says C}) = \frac{2}{3}.$$

Thus, regardless of the identity revealed by the guard, the probability that A is released is equal to 2/3, the a priori probability of being released.

Solution to Problem 1.25. Let $\overline{m}$ and $\underline{m}$ be the larger and the smaller of the two amounts, respectively. Consider the three events

$$A = \{X < \underline{m}), \qquad B = \{\underline{m} < X < \overline{m}), \qquad C = \{\overline{m} < X).$$