Part A. ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

CHAPTER 1 First-Order ODEs

Major Changes

There is more material on modeling in the text as well as in the problem set.

Some additions on population dynamics appear in Sec. 1.5.

Electric circuits are shifted to Chap. 2, where second-order ODEs will be available. This avoids repetitions that are unnecessary and practically irrelevant.

Team Projects, CAS Projects, and CAS Experiments are included in most problem sets.

SECTION 1.1. Basic Concepts. Modeling, page 2

Purpose. To give the students a first impression what an ODE is and what we mean by **solving** it.

Background Material. For the whole chapter we need integration formulas and techniques, which the student should review.

General Comments

This section should be covered relatively rapidly to get quickly to the actual solution methods in the next sections.

Equations (1)–(3) are just examples, not for solution, but the student will see that solutions of (1) and (2) can be found by calculus, and a solution $y = e^x$ of (3) by inspection.

Problem Set 1.1 will help the student with the tasks of

Solving $y' = f(x)$ by calculus

Finding particular solutions from given general solutions

Setting up an ODE for a given function as solution

Gaining a first experience in modeling, by doing one or two problems

Gaining a first impression of the importance of ODEs

without wasting time on matters that can be done much faster, once systematic methods are available.

Comment on "General Solution" and "Singular Solution"

Usage of the term "general solution" is not uniform in the literature. Some books use the term to mean a solution that includes *all* solutions, that is, both the particular and the singular ones. We do not adopt this definition for two reasons. First, it is frequently quite difficult to prove that a formula includes *all* solutions; hence, this definition of a general solution is rather useless in practice. Second, *linear* differential equations (satisfying rather general conditions on the coefficients) have no singular solutions (as mentioned in the text), so that for these equations a general solution as defined does include all solutions. For the latter reason, some books use the term "general solution" for linear equations only; but this seems very unfortunate.

SOLUTIONS TO PROBLEM SET 1.1, page 8

2. $y = -e^{-3x}/3 + c$ **4.** $y = (\sinh 4x)/4 + c$

6. Second order. **8.** First order.

10. $y = ce^{0.5x}$, $y(2) = ce = 2$, $c = 2/e$, $y = (2/e)e^{0.5x} = 0.736e^{0.5x}$

12. $y = ce^{x} + x + 1$, $y(0) = c + 1 = 3$, $c = 2$, $y = 2e^{x} + x + 1$

14. $y = c \sec x$, $y(0) = c/\cos 0 = c = \frac{1}{2}\pi$, $y = \frac{1}{2}\pi \sec x$

16. Substitution of $y = cx - c^2$ into the ODE gives

$$y'^2 - xy' + y = c^2 - xc + (cx - c^2) = 0.$$

Similarly,

$$y = \tfrac{1}{4}x^2, \qquad y' = \tfrac{1}{2}x, \qquad \text{thus} \qquad \tfrac{1}{4}x^2 - x(\tfrac{1}{2}x) + \tfrac{1}{4}x^2 = 0.$$

18. In Prob. 17 the constants of integration were set to zero. Here, by two integrations,

$$y'' = g, \qquad v = y' = gt + c_1, \qquad y = \tfrac{1}{2}gt^2 + c_1 t + c_2, \qquad y(0) = c_2 = y_0,$$

and, furthermore,

$$v(0) = c_1 = v_0, \qquad \text{hence} \qquad y = \tfrac{1}{2}gt^2 + v_0 t + y_0,$$

as claimed. Times of fall are 4.5 and 6.4 sec, from $t = \sqrt{100/4.9}$ and $\sqrt{200/4.9}$.

20. $y' = ky$. Solution $y = y_0 e^{kx}$, where y_0 is the pressure at sea level $x = 0$. Now $y(18000) = y_0 e^{k \cdot 18000} = \frac{1}{2}y_0$ (given). From this,

$$e^{k \cdot 18000} = \tfrac{1}{2}, \quad y(36000) = y_0 e^{k \cdot 2 \cdot 18000} = y_0(e^{k \cdot 18000})^2 = y_0(\tfrac{1}{2})^2 = \tfrac{1}{4}y_0.$$

22. For 1 year and annual, daily, and continuous compounding we obtain the values

$$y_a(1) = 1060.00, \qquad y_d(1) = 1000(1 + 0.06/365)^{365} = 1061.83,$$

$$y_c(1) = 1000e^{0.06} = 1061.84,$$

respectively. Similarly for 5 years,

$$y_a(5) = 1000 \cdot 1.06^5 = 1338.23, \quad y_d(5) = 1000(1 + 0.06/365)^{365 \cdot 5} = 1349.83,$$

$$y_c(5) = 1000e^{0.06 \cdot 5} = 1349.86.$$

We see that the difference between daily compounding and continuous compounding is very small.

The ODE for continuous compounding is $y_c' = ry_c$.

SECTION 1.2. Geometric Meaning of $y' = f(x, y)$. Direction Fields, page 9

Purpose. To give the student a feel for the nature of ODEs and the general behavior of fields of solutions. This amounts to a conceptual clarification before entering into formal manipulations of solution methods, the latter being restricted to relatively small—albeit important—classes of ODEs. This approach is becoming increasingly important, especially because of the graphical power of **computer software.** It is the analog of conceptual studies of the derivative and integral in calculus as opposed to formal techniques of differentiation and integration.

Comment on Isoclines

These could be omitted because students sometimes confuse them with solutions. In the computer approach to direction fields they no longer play a role.

Comment on Order of Sections

This section could equally well be presented later in Chap. 1, perhaps after one or two formal methods of solution have been studied.

SOLUTIONS TO PROBLEM SET 1.2, page 11

2. Semi-ellipse $x^2/4 + y^2/9 = 13/9$, $y > 0$. To graph it, choose the y-interval large enough, at least $0 \leqq y \leqq 4$.

4. Logistic equation (Verhulst equation; Sec. 1.5). Constant solutions $y = 0$ and $y = \frac{1}{2}$. For these, $y' = 0$. Increasing solutions for $0 < y(0) < \frac{1}{2}$, decreasing for $y(0) > \frac{1}{2}$.

6. The solution (not of interest for doing the problem) is obtained by using

$$dy/dx = 1/(dx/dy) \quad \text{and solving} \quad dx/dy = 1/(1 + \sin y) \quad \text{by integration,}$$

$$x + c = -2/(\tan \tfrac{1}{2}y + 1); \quad \text{thus} \quad y = -2 \arctan ((x + 2 + c)/(x + c)).$$

8. Linear ODE. The solution involves the error function.

12. By integration, $y = c - 1/x$.

16. The solution (not needed for doing the problem) of $y' = 1/y$ can be obtained by separating variables and using the initial condition; $y^2/2 = t + c$, $y = \sqrt{2t - 1}$.

18. The solution of this initial value problem involving the linear ODE $y' + y = t^2$ is $y = 4e^{-t} + t^2 - 2t + 2$.

20. CAS Project. (a) Verify by substitution that the general solution is $y = 1 + ce^{-x}$. Limit $y = 1$ ($y(x) = 1$ for all x), increasing for $y(0) < 1$, decreasing for $y(0) > 1$.

(b) Verify by substitution that the general solution is $x^4 + y^4 = c$. More "square-shaped," isoclines $y = kx$. Without the minus on the right you get "hyperbola-like" curves $y^4 - x^4 = const$ as solutions (verify!). The direction fields should turn out in perfect shape.

(c) The computer may be better if the isoclines are complicated; but the computer may give you nonsense even in simpler cases, for instance when $y(x)$ becomes imaginary. Much will depend on the choice of x- and y-intervals, a method of trial and error. Isoclines may be preferable if the explicit form of the ODE contains roots on the right.

SECTION 1.3. Separable ODEs. Modeling, page 12

Purpose. To familiarize the student with the first "big" method of solving ODEs, the separation of variables, and an extension of it, the reduction to separable form by a transformation of the ODE, namely, by introducing a new unknown function.

The section includes standard applications that lead to separable ODEs, namely,

1. the ODE giving $\tan x$ as solution

2. the ODE of the exponential function, having various applications, such as in radiocarbon dating

3. a mixing problem for a single tank

4. Newton's law of cooling

5. Torricelli's law of outflow.

In reducing to separability we consider

6. the transformation $u = y/x$, giving perhaps the most important reducible class of ODEs.

Ince's classical book [A11] contains many further reductions as well as a systematic theory of reduction for certain classes of ODEs.

Comment on Problem 5

From the implicit solution we can get two explicit solutions

$$y = +\sqrt{c - (6x)^2}$$

representing semi-ellipses in the upper half-plane, and

$$y = -\sqrt{c - (6x)^2}$$

representing semi-ellipses in the lower half-plane. [Similarly, we can get two explicit solutions $x(y)$ representing semi-ellipses in the left and right half-planes, respectively.] On the x-axis, the tangents to the ellipses are vertical, so that $y'(x)$ does not exist. Similarly for $x'(y)$ on the y-axis.

This also illustrates that it is natural to consider solutions of ODEs on *open* rather than on *closed* intervals.

Comment on Separability

An analytic function $f(x, y)$ in a domain D of the xy-plane can be factored in D, $f(x, y) = g(x)h(y)$, if and only if in D,

$$f_{xy}f = f_x f_y$$

[D. Scott, *American Math. Monthly* **92** (1985), 422–423]. Simple cases are easy to decide, but this may save time in cases of more complicated ODEs, some of which may perhaps be of practical interest. You may perhaps ask your students to derive such a criterion.

Comments on Application

Each of those examples can be modified in various ways, for example, by changing the application or by taking another form of the tank, so that each example characterizes a whole class of applications.

The many ODEs in the problem set, much more than one would ordinarily be willing and have the time to consider, should serve to convince the student of the practical importance of ODEs; so these are ODEs to choose from, depending on the students' interest and background.

Comment on Footnote 3

Newton conceived his method of fluxions (calculus) in 1665–1666, at the age of 22. *Philosophiae Naturalis Principia Mathematica* was his most influential work.

Leibniz invented calculus independently in 1675 and introduced notations that were essential to the rapid development in this field. His first publication on differential calculus appeared in 1684.

SOLUTIONS TO PROBLEM SET 1.3, page 18

2. $dy/y^2 = -(x + 2)\,dx$. The variables are now separated. Integration on both sides gives

$$-\frac{1}{y} = -\tfrac{1}{2}x^2 - 2x + c^*. \qquad \text{Hence} \qquad y = \frac{2}{x^2 + 4x + c}.$$

4. Set $y + 9x = v$. Then $y = v - 9x$. By substitution into the given ODE you obtain

$$y' = v' - 9 = v^2. \qquad \text{By separation,} \qquad \frac{dv}{v^2 + 9} = dx.$$

Integration gives

$$\frac{1}{3} \arctan \frac{v}{3} = x + c^*, \qquad \arctan \frac{v}{3} = 3x + c$$

and from this and substitution of $y = v - 9x$,

$$v = 3 \tan (3x + c), \qquad y = 3 \tan (3x + c) - 9x.$$

6. Set $u = y/x$. Then $y = xu$, $y' = u + xu'$. Substitution into the ODE and subtraction of u on both sides gives

$$y' = \frac{4x}{y} + \frac{y}{x} = u + xu' = \frac{4}{u} + u, \qquad xu' = \frac{4}{u}.$$

Separation of variables and replacement of u with y/x yields

$$2u\, du = \frac{8}{x}\, dx, \qquad u^2 = 8 \ln |x| + c, \qquad y^2 = x^2(8 \ln |x| + c).$$

8. $u = y/x$, $y = xu$, $y' = u + xu'$. Substitute u into the ODE, drop xu on both sides, and divide by x^2 to get

$$xy' = xu + x^2u' = \tfrac{1}{2}x^2u^2 + xu, \qquad u' = \tfrac{1}{2}u^2.$$

Separate variables, integrate, and solve algebraically for u:

$$\frac{du}{u^2} = \tfrac{1}{2}\, dx, \qquad -\frac{1}{u} = \tfrac{1}{2}(x + c^*), \qquad u = \frac{2}{c - x}.$$

Hence

$$y = xu = \frac{2x}{c - x}.$$

10. By separation, $y\, dy = -4x\, dx$. By integration, $y^2 = -4x^2 + c$. The initial condition $y(0) = 3$, applied to the last equation, gives $9 = 0 + c$. Hence $y^2 + 4x^2 = 9$.

12. Set $u = y/x$. Then $y' = u + xu'$. Divide the given ODE by x^2 and substitute u and u' into the resulting equation. This gives

$$2u(u + xu') = 3u^2 + 1.$$

Subtract $2u^2$ on both sides and separate the variables. This gives

$$2xuu' = u^2 + 1, \qquad \frac{2u\, du}{u^2 + 1} = \frac{dx}{x}.$$

Integrate, take exponents, and then take the square root:

$$\ln (u^2 + 1) = \ln |x| + c^*, \qquad u^2 + 1 = cx, \qquad u = \pm\sqrt{cx - 1}.$$

Hence

$$y = xu = \pm x\sqrt{cx - 1}.$$

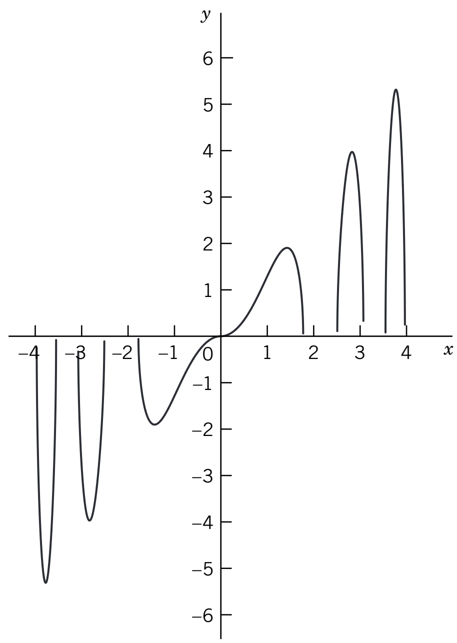
From this and the initial condition, $y(1) = \sqrt{c - 1} = 2$, $c = 5$. This gives the *answer* $y = x\sqrt{5x - 1}$.

14. Set $u = y/x$. Then $y = xu$, $y' = u + xu'$. Substitute this into the ODE, subtract u on both sides, simplify algebraically, and integrate:

$$xu' = \frac{2x^2}{u}\cos(x^2) \qquad uu' = 2x\cos(x^2), \qquad u^2/2 = \sin(x^2) + c.$$

Hence $y^2 = 2x^2(\sin(x^2) + c)$. By the initial condition, $\pi = \pi(\sin\frac{1}{2}\pi + c)$, $c = 0$,

$$y = xu = x\sqrt{2\sin(x^2)}.$$



Problem Set 1.3. Problem 14. First five real branches of the solution

16. $u = y/x$, $y = xu$, $y' = u + xu' = u + 4x^4\cos^2 u$. Simplify, separate variables, and integrate:

$$u' = 4x^3\cos^2 u, \qquad du/\cos^2 u = 4x^3\,dx, \qquad \tan u = x^4 + c.$$

Hence

$$y = xu = x\arctan(x^4 + c).$$

From the initial condition, $y(2) = 2\arctan(16 + c) = 0$, $c = -16$. *Answer:*

$$y = x\arctan(x^4 - 16).$$

18. Order terms:

$$\frac{dr}{d\theta}(1 - b\cos\theta) = br\sin\theta.$$

Separate variables and integrate:

$$\frac{dr}{r} = \frac{b\sin\theta}{1 - b\cos\theta}\,d\theta, \qquad \ln r = \ln(1 - b\cos\theta) + c^*.$$

Take exponents and use the initial condition:

$$r = c(1 - b\cos\theta), \qquad r\left(\frac{\pi}{2}\right) = c(1 - b\cdot 0) = \pi, \qquad c = \pi.$$

Hence the *answer* is $r = \pi(1 - b\cos\theta)$.

20. On the left, integrate $g(w)$ over w from y_0 to y. On the right, integrate $f(t)$ over t from x_0 to x. In Prob. 19,

$$\int_1^y we^{w^2}\,dw = \int_0^x (t - 1)\,dt.$$

22. Consider any straight line $y = ax$ through the origin. Its slope is $y/x = a$. The slope of a solution curve at a point of intersection (x, ax) is $y' = g(y/x) = g(a) = const$, independent of the point (x, y) on the straight line considered.

24. Let k_B and k_D be the constants of proportionality for the birth rate and death rate, respectively. Then $y' = k_B y - k_D y$, where $y(t)$ is the population at time t. By separating variables, integrating, and taking exponents,

$$dy/y = (k_B - k_D)\,dt, \qquad \ln y = (k_B - k_D)t + c^*, \qquad y = ce^{(k_B - k_D)t}.$$

26. The model is $y' = -Ay\ln y$ with $A > 0$. Constant solutions are obtained from $y' = 0$ when $y = 0$ and 1. Between 0 and 1 the right side is positive (since $\ln y < 0$), so that the solutions grow. For $y > 1$ we have $\ln y > 0$; hence the right side is negative, so that the solutions decrease with increasing t. It follows that $y = 1$ is stable. The general solution is obtained by separation of variables, integration, and two subsequent exponentiations:

$$dy/(y\ln y) = -A\,dt, \qquad \ln(\ln y) = -At + c^*,$$
$$\ln y = ce^{-At}, \qquad y = \exp(ce^{-At}).$$

28. The temperature of the water is decreasing exponentially according to Newton's law of cooling. The decrease during the first 30 min, call it d_1, is greater than that, d_2, during the next 30 min. Thus $d_1 > d_2 = 190 - 110 = 80$ as measured. Hence the temperature at the beginning of parking, if it had been 30 min earlier, before the arrest, would have been greater than $190 + 80 = 270$, which is impossible. Therefore Jack has no alibi.

30. The cross-sectional area A of the hole is multiplied by 4. In the particular solution, $15.00 - 0.000332t$ is changed to $15.00 - 4\cdot 0.000332t$ because the second term contains A/B. This changes the time $t = 15.00/0.000332$ when the tank is empty, to $t = 15.00/(4\cdot 0.000332)$, that is, to $t = 12.6/4 = 3.1$ hr, which is 1/4 of the original time.

32. According to the physical information given, you have

$$\Delta S = 0.15S\,\Delta\phi.$$

Now let $\Delta\phi \to 0$. This gives the ODE $dS/d\phi = 0.15S$. Separation of variables yields the general solution $S = S_0 e^{0.15\phi}$ with the arbitrary constant denoted by S_0. The angle ϕ should be so large that S equals 1000 times S_0. Hence $e^{0.15\phi} = 1000$, $\phi = (\ln 1000)/0.15 = 46 = 7.3\cdot 2\pi$, that is, eight times, which is surprisingly little. Equally remarkable is that here we see another application of the ODE $y' = ky$ and a derivation of it by a ***general principle,*** namely, by working with small quantities and then taking limits to zero.

36. B now depends on h, namely, by the Pythagorean theorem,

$$B(h) = \pi r^2 = \pi(R^2 - (R - h)^2) = \pi(2Rh - h^2).$$

Hence you can use the ODE

$$h' = -26.56(A/B)\sqrt{h}$$

in the text, with constant A as before and the new B. The latter makes the further calculations different from those in Example 5.

From the given outlet size $A = 5 \text{ cm}^2$ and $B(h)$ we obtain

$$\frac{dh}{dt} = -26.56 \cdot \frac{5}{\pi(2Rh - h^2)} \sqrt{h}.$$

Now $26.56 \cdot 5/\pi = 42.27$, so that separation of variables gives

$$(2Rh^{1/2} - h^{3/2})\, dh = -42.27\, dt.$$

By integration,

$$\tfrac{4}{3}Rh^{3/2} - \tfrac{2}{5}h^{5/2} = -42.27t + c.$$

From this and the initial condition $h(0) = R$ we obtain

$$\tfrac{4}{3}R^{5/2} - \tfrac{2}{5}R^{5/2} = 0.9333R^{5/2} = c.$$

Hence the particular solution (in implicit form) is

$$\tfrac{4}{3}Rh^{3/2} - \tfrac{2}{5}h^{5/2} = -42.27t + 0.9333R^{5/2}.$$

The tank is empty ($h = 0$) for t such that

$$0 = -42.27t + 0.9333R^{5/2}; \qquad \text{hence} \qquad t = \frac{0.9333}{42.27} R^{5/2} = 0.0221R^{5/2}.$$

For $R = 1$ m $= 100$ cm this gives

$$t = 0.0221 \cdot 100^{5/2} = 2210 \text{ [sec]} = 37 \text{ [min]}.$$

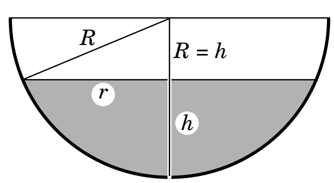
The tank has water level $R/2$ for t in the particular solution such that

$$\frac{4}{3} R \frac{R^{3/2}}{2^{3/2}} - \frac{2}{5} \frac{R^{5/2}}{2^{5/2}} = 0.9333R^{5/2} - 42.27t.$$

The left side equals $0.4007R^{5/2}$. This gives

$$t = \frac{0.4007 - 0.9333}{-42.27} R^{5/2} = 0.01260R^{5/2}.$$

For $R = 100$ this yields $t = 1260$ sec $= 21$ min. This is slightly more than half the time needed to empty the tank. This seems physically reasonable because if the water level is $R/2$, this means that 11/16 of the total water volume has flown out, and 5/16 is left—take into account that the velocity decreases monotone according to Torricelli's law.



Problem Set 1.3. Tank in Problem 36

SECTION 1.4. Exact ODEs. Integrating Factors, page 19

Purpose. This is the second "big" method in this chapter, after separation of variables, and also applies to equations that are not separable. The criterion (5) is basic. Simpler cases are solved by inspection, more involved cases by integration, as explained in the text.

Comment on Condition (5)

Condition (5) is equivalent to (6″) in Sec. 10.2, which is equivalent to (6) in the case of two variables x, y. Simple connectedness of D follows from our assumptions in Sec. 1.4. Hence the differential form is exact by Theorem 3, Sec. 10.2, part (b) and part (a), in that order.

Method of Integrating Factors

This greatly increases the usefulness of solving exact equations. It is important in itself as well as in connection with linear ODEs in the next section. Problem Set 1.4 will help the student gain skill needed in finding integrating factors. Although the method has somewhat the flavor of tricks, Theorems 1 and 2 show that at least in some cases one can proceed systematically—and one of them is precisely the case needed in the next section for *linear* ODEs.

SOLUTIONS TO PROBLEM SET 1.4, page 25

2. $(x - y)\,dx + (y - x)\,dy = 0$. Exact; the test gives -1 on both sides. Integrate $x - y$ over x:

$$u = \tfrac{1}{2}x^2 - xy + k(y).$$

Differentiate this with respect to y and compare with N:

$$u_y = -x + k' = y - x. \qquad \text{Thus} \qquad k' = y, \qquad k = \tfrac{1}{2}y^2 + c^*.$$

Answer: $\frac{1}{2}x^2 - xy + \frac{1}{2}y^2 = \frac{1}{2}(x - y)^2 = c$; thus $y = x + \tilde{c}$.

4. Exact; the test gives $e^y - e^x$ on both sides. Integrate M with respect to x to get

$$u = xe^y - ye^x + k(y).$$

Differentiate this with respect to y and equate the result to N:

$$u_y = xe^y - e^x + k' = N = xe^y - e^x.$$

Hence $k' = 0$, $k = const$. *Answer:* $xe^y - ye^x = c$.

6. Exact; the test gives $-e^x \sin y$ on both sides. Integrate M with respect to x:

$$u = e^x \cos y + k(y). \qquad \text{Differentiate:} \qquad u_y = -e^x \sin y + k'.$$

Equate this to $N = -e^x \sin y$. Hence $k' = 0$, $k = const$. *Answer:* $e^x \cos y = c$.

8. Exact; $-1/x^2 - 1/y^2$ on both sides of the equation. Integrate M with respect to x:

$$u = x^2 + \frac{x}{y} + \frac{y}{x} + k(y).$$

Differentiate this with respect to y and equate the result to N:

$$u_y = -\frac{x}{y^2} + \frac{1}{x} + k' = N, \qquad k' = 2y, \qquad k = y^2.$$

Answer:

$$x^2 + \frac{x}{y} + \frac{y}{x} + y^2 = c.$$

10. Exact; the test gives $-2x \sin (x^2)$ on both sides. Integrate N with respect to y to get

$$u = y \cos (x^2) + l(x).$$

Differentiate this with respect to x and equate the result to M:

$$u_x = -2xy \sin (x^2) + l' = M = -2xy \sin (x^2), \qquad l' = 0.$$

Answer: $y \cos (x^2) = c$.

12. Not exact. Try Theorem 1. In R you have

$$P_y - Q_x = e^{x+y} - 1 - e^{x+y}(x + 1) = -xe^{x+y} - 1 = -Q$$

so that $R = -1$, $F = e^{-x}$, and the exact ODE is

$$(e^y - ye^{-x})\, dx + (xe^y + e^{-x})\, dy = 0.$$

The test gives $e^y - e^{-x}$ on both sides of the equation. Integration of $M = FP$ with respect to x gives

$$u = xe^y + ye^{-x} + k(y).$$

Differentiate this with respect to y and equate it to $N = FQ$:

$$u_y = xe^y + e^{-x} + k' = N = xe^y + e^{-x}.$$

Hence $k' = 0$. *Answer:* $xe^y + ye^{-x} = c$.

14. Not exact; $2y \neq -y$. Try Theorem 1; namely,

$$R = (P_y - Q_x)/Q = (2y + y)/(-xy) = -3/x. \qquad \text{Hence} \qquad F = 1/x^3.$$

The exact ODE is

$$\left(x + \frac{y^2}{x^3}\right) dx - \frac{y}{x^2}\, dy = 0.$$

The test gives $2y/x^3$ on both sides of the equation. Obtain u by integrating $N = FQ$ with respect to y:

$$u = -\frac{y^2}{2x^2} + l(x). \qquad \text{Thus} \qquad u_x = \frac{y^2}{x^3} + l' = M = x + \frac{y^2}{x^3}.$$

Hence $l' = x$, $l = x^2/2$, $-y^2/2x^2 + x^2/2 = c^*$. Multiply by 2 and use the initial condition $y(2) = 1$:

$$x^2 - \frac{y^2}{x^2} = c = 3.75$$

because inserting $y(2) = 1$ into the last equation gives $4 - 0.25 = 3.75$.

16. The given ODE is exact and can be written as $d(\cos xy) = 0$; hence $\cos xy = c$, or you can solve it for y by the usual procedure. $y(1) = \pi$ gives $-1 = c$. *Answer:* $\cos xy = -1$.

18. Try Theorem 2. You have

$$R^* = (Q_x - P_y)/P = \left[\frac{1}{y} \cos xy - x \sin xy - \left(-x \sin xy - \frac{x}{y^2}\right)\right] \Big/ P = \frac{1}{y}.$$

Hence $F^* = y$. This gives the exact ODE

$$(y \cos xy + x)\, dx + (y + x \cos xy)\, dy = 0.$$

In the test, both sides of the equation are $\cos xy - xy \sin xy$. Integrate M with respect to x:

$$u = \sin xy + \tfrac{1}{2}x^2 + k(y). \qquad \text{Hence} \qquad u_y = x \cos xy + k'(y).$$

Equate the last equation to $N = y + x \cos xy$. This shows that $k' = y$; hence $k = y^2/2$. *Answer:* $\sin xy + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 = c$.

20. Not exact; try Theorem 2:

$$\begin{aligned} R^* = (Q_x - P_y)/P &= [1 - (\cos^2 y - \sin^2 y - 2x \cos y \sin y)]/P \\ &= [2 \sin^2 y + 2x \cos y \sin y]/P \\ &= 2(\sin y)(\sin y + x \cos y)/(\sin y \cos y + x \cos^2 y) \\ &= 2(\sin y)/\cos y = 2 \tan y. \end{aligned}$$

Integration with respect to y gives $-2 \ln(\cos y) = \ln(1/\cos^2 y)$; hence $F^* = 1/\cos^2 y$. The resulting exact equation is

$$(\tan y + x)\, dx + \frac{x}{\cos^2 y}\, dy = 0.$$

The exactness test gives $1/\cos^2 y$ on both sides. Integration of M with respect to x yields

$$u = x \tan y + \tfrac{1}{2}x^2 + k(y). \qquad \text{From this,} \qquad u_y = \frac{x}{\cos^2 y} + k'.$$

Equate this to $N = x/\cos^2 y$ to see that $k' = 0$, $k = const.$ *Answer:* $x \tan y + \tfrac{1}{2}x^2 = c$.

22. **(a)** Not exact. Theorem 2 applies and gives $F^* = 1/y$ from

$$R^* = (Q_x - P_y)/P = (0 - \cos x)/(y \cos x) = -\frac{1}{y}.$$

Integrating M in the resulting exact ODE

$$\cos x\, dx + \frac{1}{y^2}\, dy = 0$$

with respect to x gives

$$u = \sin x + k(y). \qquad \text{From this,} \qquad u_y = k' = N = \frac{1}{y^2}.$$

Hence $k = -1/y$. *Answer:* $\sin x - 1/y = c$.

Note that the integrating factor $1/y$ could have been found by inspection and by the fact that an ODE of the general form

$$f(x)\, dx + g(y)\, dy = 0$$

is always exact, the test resulting in 0 on both sides.

(b) Yes. Separation of variables gives

$$dy/y^2 = -\cos x\, dx. \qquad \text{By integration,} \qquad -1/y = -\sin x + c^*$$

in agreement with the solution in (a).

(d) seems better than (c). But this may depend on your CAS. In (d) the CAS may draw vertical asymptotes that disturb the figure.

From the solution in (a) or (b) the student should conclude that for each nonzero $y(x_0) = y_0$ there is a unique particular solution because

$$\sin x_0 - 1/y_0 = c.$$